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HENRI BROCARD AND THE GEOMETRY OF THE TRIANGLE.

By Laura Guggenbuhl.

During the last half of the nineteenth century there was a great revival of interest in the field of geometry. A large part of this interest had its origin in a simple problem submitted to a contemporary mathematical periodical by a French army officer. The problem was to find a point O within a triangle ABC such that the angles OAB, OBC and OCA would be equal. The name of the army captain who submitted the problem was Brocard—Pierre René Jean-Baptiste Henri Brocard.

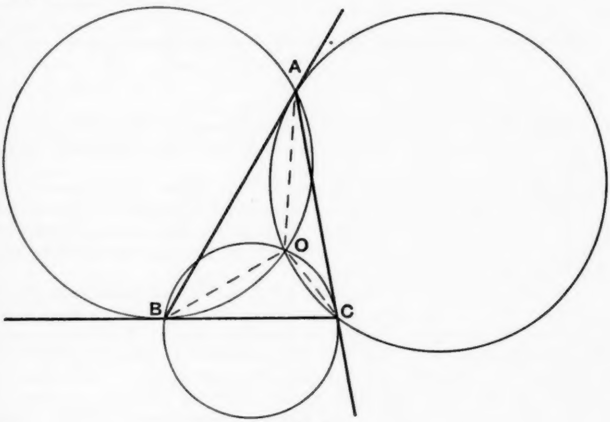


Fig. 1.

Soon thereafter many different solutions of the problem were published; and gradually a rather elaborate unit of geometric theory had developed from this source. A solution of the problem is readily accessible in a text book on

Modern Geometry, and nothing more than a brief summary will be given here. The most picturesque solution is one in which circles are drawn as follows—circles tangent to the side AB of the triangle ABC, at the vertex A, and at the vertex B respectively, and passing through the vertex C; and four other similar circles. Three of these six circles are concurrent at a point O, and the other three at a point O'. The points O and O' satisfy the conditions of the above problem, and are called the Brocard points of the triangle.

The angle OAB (angle ω) is called the Brocard angle of triangle ABC. It is a simple matter to prove from the following diagram, in which construction lines can be easily identified, that

$$\cot \omega = \cot A + \cot B + \cot C.$$

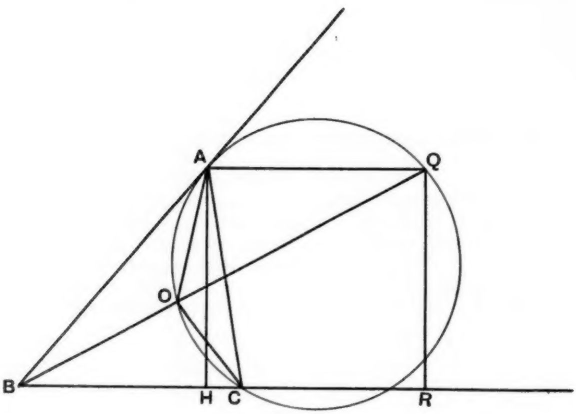


FIG. 2.

The Brocard circle of a triangle is the circle drawn on PK as diameter, where P is the circumcenter of the triangle and K is its symmedian point. It passes through O and O'.

Around 1890 several books were published, which were devoted to the theory of the Brocard configuration. However, biographical details about Brocard remained relatively obscure. In English, such details were virtually non-existent. It is the purpose of this article to fill an often expressed wish to know a little more about the man.

Henri Brocard was born on May 12, 1845, at Vignot (Meuse) in the north-eastern corner of France. He was a son of Jean Sebastion and Elizabeth Auguste Liouville Brocard. No record has been found of brothers or sisters, and Brocard never married. In 1894 he published an autobiography, giving the most minute details, for the first fifty years of his life. This account tells of his education at the Lycée of Marseille, and the Lycée and Académie of Strasbourg. He attended the Ecole Polytechnique from 1865 to 1867, and then became a member of the Engineers of the French Army. His army career was not one of active military combat. It is known that he was a prisoner of war at Sedan in 1870; but for the most part, his activities in the army were devoted to organizing courses in physics and chemistry at various military colleges, and to research, particularly in meteorology and mathe-

matics. For several years of his army service he was assigned to North Africa, and he has been called the co-founder of the Meteorological Institute of Algiers.

Brocard became a member of the Society of Letters, Sciences and Arts of Bar-le-Duc in 1894. It is through the pages of the publications of this Society that one can follow the activities of the last twenty-six years of his life. Though he declined the honour of becoming President, he obviously found great happiness in his work as Librarian of the Society. He became interested in local affairs, and devoted himself to public service. Largely through his efforts, one of the streets of Bar-le-Duc was named in honour of Louis Joblot, a native Barrisien, who was an acknowledged but almost forgotten pioneer in the field of microscopy. When he retired from the army in 1910, Brocard was a lieutenant-colonel, and an Officer in the Legion of Honor. In the years of his retirement he spent much of his time in the pleasant garden in the rear of his house, at 75 Rue de Ducs in Bar-le-Duc, where he would use a small telescope for meteorological and astronomical observations. In spite of the fact that he lived completely alone and rarely had visitors, he always attended the quadrennial meetings of the International Congress of Mathematicians. On January 16, 1922, he was found dead at his desk, and in accordance with his specific request, he was buried in the cemetery at Vignot, next to his father and mother.

His most colourful and ambitious publication was *Notes de Bibliographie des Courbes Géométriques*. Volume I appeared in 1897 and Volume II in 1899. Probably no more than about fifty copies of this work were prepared, and it has become exceedingly rare and valuable. It was lithographed in the print-script of the author, and it was privately distributed. Although it was described as being neither a dictionary nor an encyclopaedia, it may be regarded as a source book of geometric curves. It has a painstakingly carefully prepared index of over one thousand numbered and named curves. About twenty years after the appearance of this work, virtually the same material appeared in printed form under the joint authorship of H. Brocard and T. Lemoyne.

Brocard has not yet found a place, nor even a line, in books on the history of mathematics ; for his was obviously a most modest personality, and he had the humility of the true scholar. However, his influence upon his contemporaries was such that his name has been used to identify not only the Brocard points of a triangle, and the Brocard circle (which he discovered), but also many other geometric conclusions.

Perhaps some day, you too will be passing through Bar-le-Duc. If so, you might enjoy a leisurely stroll along the Quai Carnot to the municipal library. The present Librarian of the Society of Letters, Sciences and Arts is M. Rogie, and the President is M. Lucien Braye. If you are interested in such things, M. Rogie and M. Braye would surely be more than pleased to place before you with reverent pride, the Library's treasured copy of Brocard's *Notes de Bibliographie des Courbes Géométriques*. L. G.

GLEANINGS FAR AND NEAR.

1751. I went early in the evening to Mr. Barclay's at Moreham, a good sensible man but with not many words or topics of conversation, for he was a great mathematician : with the help of his wife and daughter, however, we made shift to spend the evening.—*The Autobiography of Dr. Alexander Carlyle of Inveresk, 1722–1805*, edited by John Hill Burton, new edition, Foulis, 1910. [Per Mr. James Buchanan.]

TWO INEQUALITIES*

By G. N. Watson.

(I). The following inequality is a straight generalisation of one of the most important inequalities occurring in elementary analysis. It is consequently of some intrinsic interest, even though it has to do with a determinant.

Let n be a positive integer ($\geqslant 2$) *and let $a, b, \ldots, h$ be n real numbers* (*unrestricted as to sign*) *arranged in descending order of magnitude, and no two being equal. Let x be a positive number, which will be regarded as variable. Let the determinant*

$$\begin{vmatrix} x^a & & x^b & \ldots & x^h \\ a^{n-2} & & b^{n-2} & \ldots & h^{n-2} \\ a^{n-3} & & b^{n-3} & \ldots & h^{n-3} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a & & b & \ldots & h \\ 1 & & 1 & \ldots & 1 \end{vmatrix}$$

be denoted by $D_n(x\,;\,a, b, \ldots, h)$ or briefly (*when there is no risk of confusion*) *by $D_n(x)$. Then* (i) *$D_n(x)$ is* positive *when $x > 1$; and* (ii) *$D_n(x)$ has the same sign as $(-1)^{n-1}$ when $0 < x < 1$.*

These results are obvious when $n = 2$; for larger values of n, we prove them by induction. We suppose that they hold for $D_n(x\,;\,a, b, \ldots, h)$, and we consider $D_{n+1}(x\,;\,a, b, \ldots, h, k)$ with $k < h$. By performing the differentiations, we easily see that

$$x^{k+1} \frac{d}{dx} \left(x^{-k} D_{n+1}(x\,;\,a, b, \ldots, h, k) \right)$$

is expressible as a determinant which differs from $D_{n+1}(x)$ merely in having the elements of the first row replaced by

$$(a-k)\,x^a, \quad (b-k)\,x^b, \ldots, (h-k)\,x^h, \; 0.$$

In this determinant, we multiply the elements of the third and all subsequent rows by k and then subtract the results from the corresponding elements of each preceding row. We thus obtain a determinant in which the elements of the first column are

$$(a-k)\,x^a, \; (a-k)\,a^{n-2}, \; (a-k)\,a^{n-3}, \ldots, a-k, \; 1\,;$$

and similarly for all the other columns except the last; the last column has the elements $0, 0, 0, \ldots, 0, 1$.

This determinant is evidently equal to the cofactor of the last element of its last row; from the elements in the various columns of this cofactor we may remove the factors $a-k$, $b-k$, $\ldots$, $h-k$, and we then see that

* [The *Gazette* for December 1903 contained a modest mathematical note which described " A method of determining a very rapidly convergent series for the square root of any positive integer H "; the claim put forward for the method was not without warrant, because it was remarked that eight terms of the series thus obtained for $\sqrt{2}$ yielded its value to 390 places of decimals. This note, from one of the pupils of F. S. Macaulay at St. Paul's School, was the precursor of a long series of publications in many of the leading mathematical periodicals of the world, establishing their author as a master of all the resources of modern analysis. We are deeply grateful to Professor Watson for allowing us to celebrate his mathematical jubilee by publishing his characteristic communication " Two inequalities " just fifty years to the month after his first appearance as a creative mathematician. Ed.]

$$x^{k+1}\frac{d}{dx}\left(x^{-k}D_{n+1}(x\,;\,a,\,b,\,\dots,\,h,\,k)\right)$$
$$=(a-k)(b-k)\dots(h-k)\,D_n(x\,;\,a,\,b,\,\dots\,,\,h).$$

Now $D_n(x)$ is *ex hypothesi* positive when $x>1$; accordingly this result shews that $x^{-k}D_{n+1}(x)$ is an increasing function of x when $x>1$, and this function evidently vanishes when $x=1$. Consequently $x^{-k}D_{n+1}(x)$ is positive when $x>1$, and the required induction is established for the range $x>1$. The reader should now have no difficulty in establishing the induction for the range $0<x<1$ in a similar manner.

There is also no difficulty in constructing a modified form of the inequality which holds when we remove the limitation that $a, b, \dots, h$ are to be in descending order of magnitude; the limitation that they are to be unequal must, of course, be retained.

I have not traced the general inequality in the treatise *Inequalities* (1934) by Hardy, Littlewood and Pólya; the special case of it in which all the quotients of differences of the type $a-b, \dots, a-h, \dots, b-h, \dots$ are rational numbers is, however, easily deducible from a set of results given by H. W. Segar some sixty years ago in several papers in the *Messenger of Mathematics*.

In the special case $n=3$ we have

$$(b-c)\,x^a+(c-a)\,x^b+(a-b)\,x^c>0, \quad (x\neq 1)$$

and from this we deduce the standard inequality that

$$x^a-1>a\,(x-1)$$

when $a>1$, by taking $b=1$, $c=0$. Conversely, the former inequality is derivable from the latter by an appropriate change of variables.

(II.) *Schur's inequality*. The theorem that, *if $\mu\geqslant 0$ and x, y, z are all positive, then*

$$f(x,y,z\,;\,\mu)\equiv x^\mu(x-y)(x-z)+y^\mu(y-z)(y-x)+z^\mu(z-x)(z-y)\geqslant 0$$

with equality occurring when and only when $x=y=z$, is given by Hardy, Littlewood and Pólya, *Inequalities* (1934), 64.

When this result attracted my attention, I found it an easy matter to construct a proof as follows: the symmetry of $f(x, y, z\,;\,\mu)$ in x, y, z means that there is no loss of generality in assuming that $x\geqslant y\geqslant z$. Now

$$f(x,y,z\,;\,\mu)\equiv(x^\mu-y^\mu)(x-y)(x-z)+y^\mu(x-y)^2+z^\mu(x-z)(x-y)\,;$$

and, with the assumption just stated, each of the three terms on the right is either positive or zero.

I also noticed that, since

$$f(x,y,z\,;\,\mu)\equiv xyzf(1/x,\,1/y,\,1/z\,;\,-\mu-1),$$

the inequality also holds for all values of μ for which $\mu\leqslant -1$.

Having gone so far, I wrote and asked Prof. Hardy whether he could tell me anything more about the theorem; in reply, he sent me a proof equivalent to the proof just given (which he described as regrettably unsymmetrical) with the information that he had received the inequality in a letter from Prof. Schur, and that, so far as he knew, the result had not previously been published.

I have subsequently noticed that the proof of the inequality and the

extension to negative values of $\mu + 1$ have been published by S. Barnard and J. M. Child, *Higher Algebra* (1936), 217.

It seems, however, that no previous writer has observed that Schur's inequality holds for the range $-1 < \mu < 0$; and, what is much more remarkable, for this range it is possible to construct a proof in which x, y, z enter symmetrically.

Evidently

$$y^\mu (y-z)(y-x) + z^\mu (z-x)(z-y) \equiv (y^{\mu+1} - z^{\mu+1})(y-z) + x(z^\mu - y^\mu)(y-z);$$

and so, by summing, we have

$$2f(x, y, z;\ \mu) = \Sigma (y^{\mu+1} - z^{\mu+1})(y-z) + \Sigma x (z^\mu - y^\mu)(y-z).$$

Whatever be the relative values of x, y, z, all of the expressions

$$(y^{\mu+1} - z^{\mu+1})(y-z), \quad x(z^\mu - y^\mu)(y-z)$$

and the expressions obtained from them by cyclical changes of x, y, z are non-negative for $-1 \leqslant \mu \leqslant 0$; and the required result is now obvious.

It is not within my power to give a symmetrical proof of the inequality for general positive values of μ, but I give symmetrical proofs for $\mu = 1$ and for $\mu = 2$. For $\mu = 2$ we have the identity

$$2f(x, y, z;\ 2) \equiv \Sigma (y-z)^2 (y+z-x)^2,$$

as the reader will easily verify. This establishes Schur's inequality for $\mu = 2$, since each of the three terms of the sum on the right is obviously non-negative.

In like manner, Schur's inequality for $\mu = 1$ follows from the identity

$$2\Sigma (x^2 + yz) . f(x, y, z;\ 1) \equiv (\Sigma x) \Sigma (y-z)^2 (x-y-z)^2 + xyz \Sigma (2x-y-z)^2,$$

this being the simplest symmetrical identity known to me which establishes the desired result. The first step towards obtaining this identity is to write $t(y+z)$ for x in $f(x, y, z;\ 1)$ and to rearrange the result in the form

$$f(x, y, z;\ 1) \equiv (y+z)\{(t+1)(t-1)^2 (y-z)^2 + (2t-1)^2 tyz\},$$

and then to substitute $x/(y+z)$ for t.

A certain amount of perseverance is necessary to check the correctness of the identity by verifying that the expressions on the left and on the right are both equal to

$$2\Sigma x^5 - 2\Sigma x^3 (y^2 + z^2) + 2xyz (2\Sigma x^2 - \Sigma yz).$$

G. N. W.

1752. In recent years the gullible public have been well primed with the propaganda that not only is jazz old-fashioned but that swing is also outdated. The new music, rebop, bebop, or just plain bop, was hailed by the full spate of New York publicity—which the London counterpart meekly followed. The radio and the juke-boxes joined in the fray and before swing could turn round everybody was demanding bop, without anybody knowing in the slightest degree what it was all about. Characterized by meaningless displays of grotesque techniques and mathematical chord and harmonic progressions, it appeals only to the analytical musical mind and evokes about the same amount of emotional pleasure as a Euclid theorem. Without melody, without any logical thematic development, it has been supported only by those who allow the musical journalists to do their thinking for them.—Rex Harris, *Jazz* (Penguin Books, 1952), p. 172. [Per Dr. S. Rushton.]

PERSPECTIVE TRIADS.

By W. R. Andress and W. Saddler, with a note by W. W. Sawyer.

1. Consider two triads of points, a, b, c, a', b', c' in a plane ; then, using homogeneous coordinates and regarding the points as specified by the corresponding vectors, we may write

$$\left.\begin{aligned} a' &= l_1 a + m_1 b + n_1 c,\\ b' &= l_2 a + m_2 b + n_2 c,\\ c' &= l_3 a + m_3 b + n_3 c,\end{aligned}\right\} \qquad (1)$$

and writing $b\times c=\alpha$, $c\times a=\beta$, $a\times b=\gamma$, then $a\times a'=m_1\gamma-n_1\beta$, $b\times b'=n_2\alpha-l_2\gamma$, $c\times c'=l_3\beta-m_3\alpha$. Hence the lines aa', bb', cc' are concurrent provided

$$\begin{vmatrix} 0 & -n_1 & m_1\\ n_2 & 0 & -l_2\\ -m_3 & l_3 & 0\end{vmatrix}=0,$$

that is, provided

$$l_2 m_3 n_1 = l_3 m_1 n_2. \qquad (2)$$

If we rotate the suffixes 1, 2, 3 cyclically, then the lines ab', bc', ca' are concurrent provided

$$l_3 m_1 n_2 = l_1 m_2 n_3, \qquad (3)$$

so that combining (2) and (3),

$$l_2 m_3 n_1 = l_3 m_1 n_2 = l_1 m_2 n_3, \qquad (4)$$

which is the condition that ac', ba', cb' are concurrent. Hence if two triads are in double perspective, in this way, they are also in triple perspective.†

If the order of the points a', b', c' is maintained in the cyclic rotation we shall say that the triads a', b', c', a, b, c, are in *direct* triple perspective, whilst if the triads c', b', a' ; a, b, c are in direct triple perspective, then a', b', c', a, b, c are said to be in *reverse* triple perspective ; and the condition for this is clearly

$$l_1 n_2 m_3 = l_2 n_3 m_1 = l_3 n_1 m_2. \qquad (5)$$

If further the two triads a', b', c', a, b, c are in both direct and reverse triple perspective they are then said to be in sextuple perspective. If the two triads are in either direct or reverse triple perspective and if also one other set of joins is concurrent (that is, one equation of (4) or (5) is additionally satisfied) the triads are then said to be in quadruple perspective.

2. We show next that by suitably re-weighting the points a, b, c, a', b', c' the linear transformation ϕ giving direct triply perspective triads can be reduced to a canonical form ψ involving three parameters p, q, r. Writing A, A' for the matrices whose vector components are the vectors representing the triads a, b, c ; a', b', c' respectively, then

$$A'=\phi A,$$

where

$$\phi=\begin{bmatrix} l_1 & m_1 & n_1\\ l_2 & m_2 & n_2\\ l_3 & m_3 & n_3\end{bmatrix}.$$

Now re-weighting a', b', c', a, b, c, is merely equivalent to premultiplying and

† *See* Forder, *Calculus of extension*, p. 60.

postmultiplying ϕ by diagonal matrices, say D_1 and D_2 respectively, and it follows easily by taking

$$D_1 = \operatorname{diag}[(l_1 m_1 n_1)^{-\frac{1}{3}}, \quad (l_2 m_2 n_2)^{-\frac{1}{3}}, \quad (l_3 m_3 n_3)^{-\frac{1}{3}}]$$
$$D_2 = \operatorname{diag}[(l_1 l_2 l_3)^{-\frac{1}{3}}, \quad (m_1 m_2 m_3)^{-\frac{1}{3}}, \quad (n_1 n_2 n_3)^{-\frac{1}{3}}]$$

that

$$D_1 \phi D_2 = \begin{bmatrix} \left(\dfrac{l_1}{m_1 n_1 l_2 l_3}\right)^{\frac{1}{3}} & \left(\dfrac{m_1}{n_1 l_1 m_2 m_3}\right)^{\frac{1}{3}} & \left(\dfrac{n_1}{l_1 m_1 n_2 n_3}\right)^{\frac{1}{3}} \\ \left(\dfrac{l_2}{m_2 n_2 l_3 l_1}\right)^{\frac{1}{3}} & \left(\dfrac{m_2}{n_2 l_2 m_3 m_1}\right)^{\frac{1}{3}} & \left(\dfrac{n_2}{l_2 m_2 n_3 n_1}\right)^{\frac{1}{3}} \\ \left(\dfrac{l_3}{m_3 n_3 l_1 l_2}\right)^{\frac{1}{3}} & \left(\dfrac{m_3}{n_3 l_3 m_1 m_2}\right)^{\frac{1}{3}} & \left(\dfrac{n_3}{l_3 m_3 n_1 n_2}\right)^{\frac{1}{3}} \end{bmatrix}$$

and hence writing

$$l_1 m_2 n_3 = l_2 m_3 n_1 = l_3 m_1 n_2 = \lambda^3$$
$$l_1 m_3 n_2 = p^3, \quad l_2 m_1 n_3 = q^3, \quad l_3 m_2 n_1 = r^3$$

the matrix reduces immediately to

$$D_1 \phi D_2 = \lambda^{-2} \psi,$$

where ψ takes the canonical persymmetric form

$$\psi = \begin{bmatrix} p & q & r \\ q & r & p \\ r & p & q \end{bmatrix}$$

so that taking a, b, c as base points the general triad a', b', c' in triple perspective with a, b, c, may be written in the form

$$a' = pa + qb + rc,$$
$$b' = qa + rb + pc,$$
$$c' = ra + pb + qc.$$

Similarly, or by interchanging b', c', for reverse triply perspective triads, the matrix ϕ may be reduced to the form

$$\bar{\psi} = \begin{bmatrix} p & q & r \\ r & p & q \\ q & r & p \end{bmatrix}$$

so that if the triad A' is in reverse triple perspective with the triad A, then $A' = \bar{\psi} A$.

If further the two triads a', b', c', a, b, c are in sextuple perspective, then by (5) we must also have $p^3 = q^3 = r^3$, but since a', b', c' are to be distinct and

$$|\psi| = p^3 + q^3 + r^3 - 3pqr \neq 0,$$

so that

$$p + q + r \neq 0, \quad p + \omega q + \omega^2 r \neq 0, \quad p + \omega^2 q + \omega r \neq 0,$$

where $\omega^3 = 1$, it follows that ψ, with possible re-weighting, is reducible to the forms θ, $\bar{\theta}$ where

$$\theta = \begin{bmatrix} \omega & 1 & 1 \\ 1 & \omega & 1 \\ 1 & 1 & \omega \end{bmatrix}, \qquad \bar{\theta} = \begin{bmatrix} \omega^2 & 1 & 1 \\ 1 & \omega^2 & 1 \\ 1 & 1 & \omega^2 \end{bmatrix}$$

or, by further re-weighting to make the elements of the first column and row each unity, ψ reduces to the forms Ω, $\bar{\Omega}$ where

$$\Omega\sqrt{3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}, \qquad \bar{\Omega}\sqrt{3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}$$

in which it is clear that Ω, $\bar{\Omega}$ only differ in as much as they interchange b', c'. Hence, if D_1, D_2 are diagonal matrices, all triads in sextuple perspective with a, b, c are given by

$$A' = D_1 \Omega D_2 A,$$

or if the weighting of A and A' be suitably chosen,

$$A' = \Omega A.$$

The factor $\sqrt{3}$ was introduced so that $\Omega\bar{\Omega} = I$ and hence Ω is a unitary matrix, and further $\Omega^2 = \bar{\Omega}^2 = I'$, where I' is the permutation matrix

$$I' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Also since $I'\psi = \bar{\psi}$, we have $\Omega\Omega\psi = \bar{\psi}$ and hence $\Omega\psi = \bar{\Omega}\bar{\psi}$.

3. *The latent vectors of ψ and $\bar{\psi}$.*

Since

$$\bar{\psi} = \begin{bmatrix} p & q & r \\ r & p & q \\ q & p & r \end{bmatrix}$$

we have

$$\begin{aligned} |\, \bar{\psi} - \lambda I \,| &= (p-\lambda)^3 + q^3 + r^3 - (p - \lambda)qr \\ &= (p - \lambda + q + r)(p - \lambda + \omega q + \omega^2 r)(p - \lambda + \omega^2 q + \omega r). \end{aligned}$$

Hence the latent roots are

$$\lambda_1 = p + q + r, \quad \lambda_2 = p + \omega q + \omega^2 r, \quad \lambda_3 = p + \omega^2 q + \omega r.$$

Using these roots in turn the corresponding latent vectors are, neglecting scalar factors $(1, 1, 1)$, $(1, \omega, \omega^2)$, $(1, \omega^2, \omega)$ which are the three vector components of Ω and $\bar{\Omega}$, and it is immediately verified that

$$\bar{\Omega}\bar{\psi}\Omega = \text{diag}\,[\lambda_1, \lambda_2, \lambda_3].$$

By interchanging the second and third vector components of Ω it is clear that ψ has the same latent vectors as $\bar{\psi}$, and since $\bar{\psi}\Omega = \psi\bar{\Omega}$ we may write the preceding equation in the form $\bar{\Omega}\psi\bar{\Omega} = \text{diag}\,[\lambda_1, \lambda_2, \lambda_3]$.

We may write the latent vectors as

$$a + b + c = h, \quad a + \omega b + \omega^2 c = h_1, \quad a + \omega^2 b + \omega c = h_2, \quad \ldots\ldots\ldots\ldots\ldots(6)$$

or if H is a matrix with vector components h, h_1, h_2, then $\Omega A = H$ and since $\omega^3 = 1$, it should be noticed that the triads a, b, c, h, h_1, h_2 are in sextuple perspective.

Theorem. If any two triads are in triple perspective then there exists a triad with which they are both in sextuple perspective.

This follows directly; for if the triads are A', A where $A' = \psi A$, then in this transformation the triad H of the latent vectors remains invariant, and since

H, A are in sextuple perspective, so also are the triads H, A', since perspective properties are unaffected by the transformation.

Theorem. If two triads A_1, A_2 are each in sextuple perspective with a third triad, then A_1, A_2 are themselves in triple perspective.

If the triads A_1, A_2 are suitably ordered and if D_1, D_2 represent diagonal matrices then we may take

$$A_1 = \Omega D_1 A, \quad A_2 = \Omega D_2 A$$

whence

$$A = D_2^{-1}\Omega A_2$$

and

$$A_1 = \Omega D_1 D_2^{-1}\bar{\Omega} A = \Omega D_3 \bar{\Omega} A_2.$$

But $\Omega D_3 \bar{\Omega} = \bar{\psi}$ and hence A_1 A_2 are themselves in triple perspective.

4. *Relations between ψ and $\bar{\psi}$.*

In this section only, we use D to typify a diagonal matrix, and ψ, $\bar{\psi}$ to typify the matrices corresponding to the transformations for direct and for reverse triple perspective, and the following relations in this sense are merely diagrammatic. (a) Since $\bar{\Omega}$, Ω are the latent vectors of ψ, $\bar{\psi}$ we have

$$\psi = \Omega D \Omega = \bar{\Omega} D \bar{\Omega},$$

whence

$$\psi^{-1} = \Omega^{-1} D^{-1} \Omega^{-1} = \bar{\Omega} D \bar{\Omega} = \psi.$$

Hence as is obvious, if the triads A', A are in triple perspective, so also are the triads A, A'; but further, since ψ is non-singular, ψ^{-1} is equivalent to the adjoint of ψ and hence if the triads A', A are in triple perspective, then dually, so also are the corresponding sides $\alpha'\beta'\gamma'$, $\alpha\beta\gamma$ of the triangles $a'b'c'$, abc (that is, the meets of the lines $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$ are collinear, and the three such lines are concurrent).

(b) Again,

$$\psi\psi = \Omega D \Omega \bar{\Omega} D \bar{\Omega} = \Omega D \bar{\Omega} = \bar{\psi},$$

and similarly,

$$\bar{\psi}\bar{\psi} = \bar{\psi}, \quad \psi\bar{\psi} = \bar{\psi}\psi = \bar{\psi},$$

so that varying the ratios $p : q : r$ but fixing the weights of a, b, c we have the theorem: there exists an ∞^2 of triads each in triple perspective with a given triad a, b, c and which are themselves mutually in triple perspective.

5. *The centres of perspective.*

(a) For triply perspective triads A, A' we have $A' = \psi A$, that is,

$$a' = pa + qb + rc,$$
$$b' = qa + rb + pc,$$
$$c' = ra + pb + qc,$$

whence

$$[aa' \,.\, bb'] = (abb')a' - (a'bb')a$$
$$= pqr(a/p + b/q + c/r),$$

and hence permuting the letters p, q, r cyclically, the three centres of perspective $C(c_1, c_2, c_3)$ are given by

$$c_1 = a/p + b/q + c/r,$$
$$c_2 = a/q + b/r + c/p,$$
$$c_3 = a/r + b/p + c/q,$$

or

$$C = \psi^* A,$$

where

$$\psi^* = \begin{bmatrix} 1/p & 1/q & 1/r \\ 1/q & 1/r & 1/p \\ 1/r & 1/p & 1/q \end{bmatrix}$$

The form of ψ^* shows immediately that the triad C is in triple perspective with the triad A,† and hence also with the triad A'.

(*b*) Consider next the centres of perspective for two sextuply perspective triads A, A'. Since the triads A, A' are in both direct and reverse triple perspective there will be two triads C, $\bar{C}$ respectively which are the centres for the direct and reverse triple perspectives.

If $D = \text{diag}\,[\lambda, \mu, \nu]$, the sextuple perspective can be given by $A' = \Omega D A$, and calculating C, $\bar{C}$ as in the last paragraph (*a*) we obtain immediately

$$C = \begin{bmatrix} \omega\lambda & \mu & \nu \\ \lambda & \omega\mu & \nu \\ \lambda & \mu & \omega\nu \end{bmatrix} A = \theta D A.$$

Reversing the cyclic order of a', b', c'.

$$\bar{C} = \bar{\theta} D A,$$

where θ, $\bar{\theta}$ are the matrices introduced in § 2.

Theorem. If the triads A, A' are in sextuple perspective, with the triad C as a triad of centres of perspective, then the triads C, A are themselves in sextuple perspective.

This follows immediately from the form of the matrix θD. Interchanging the triads A, A', then the triads C, A' are also in sextuple perspective, and so also are the pairs of triads A, $\bar{C}$; A', $\bar{C}$.

Theorem. If C, $\bar{C}$ are the two triads of the centres of perspective of the sextuply perspective triads A, A', then C, $\bar{C}$ are themselves in sextuple perspective.

This follows by noting that, neglecting scalar factors, $\theta^2 = \bar{\theta} = \theta^{-1}$ and since $C = \theta D A$, $\bar{C} = \bar{\theta} D A$, whence $\bar{C} = \bar{\theta}\theta^{-1} C = \theta C$, and C, $\bar{C}$ are in sextuple perspective. In fact the four triads A, A', C, $\bar{C}$, taken in pairs, are mutually in sextuple perspective.

6. *Theorem.* If two triads C, $\bar{C}$ are the centres of perspective of the sextuply perspective triads A, A', then (*a*) the triads A', $\bar{C}$ are the centres of perspective for the triads A, C; and (*b*) the triads A, A' are the centres of perspective for the triads C, $\bar{C}$. Thus the four triads A, A', C, C form a closed set in as much as the centres of perspective derived from any two of the triads are the remaining two triads.

First we have a lemma concerning the relations between θ and Ω. It was mentioned in § 2 that by re-weighting the points A, A' the matrices θ, Ω were equivalent. More precisely, if we introduce diagonal matrices

$$\alpha = \text{diag}\,[\omega, 1, 1], \quad \bar{\alpha} = \text{diag}\,[\omega^2, 1, 1]$$

then immediately

$$\alpha\bar{\theta}\alpha = \omega\Omega, \quad \bar{\alpha}\theta\bar{\alpha} = \omega^2\bar{\Omega}, \quad \alpha\bar{\alpha} = I,$$

whence

$$\bar{\alpha}\Omega\bar{\alpha} = \omega^2\bar{\theta} \quad \text{and} \quad \alpha\bar{\Omega}\alpha = \omega\theta,$$

and

$$\theta\alpha\theta = \omega\bar{\alpha}I' = \omega I'\bar{\alpha}.$$

† Jahnke, *Crelle*, 123 (1901), p. 42. *See also* Forder, *loc. cit.*

(*a*) Now let C^* and $\bar{C}^*$ be the centres of perspective for the triads A, C. Then neglecting scalar factors throughout,

$$C = \theta A = \alpha \bar{\Omega} \alpha A \quad \text{or} \quad \bar{\alpha} C = \bar{\Omega}(\alpha A)$$

and hence the centres C^*, $\bar{C}^*$ are given by

$$C^* = \bar{\theta} \alpha A, \quad \bar{C}^* = \theta \alpha A.$$

So
$$\alpha C^* = \alpha \bar{\theta} \alpha A = \Omega A = A'$$

and
$$\alpha \bar{C}^* = \alpha \theta \alpha A = \bar{\theta} \theta \alpha \theta \alpha A$$
$$= \bar{\theta} I' A = I' \bar{\theta} A = I' \bar{C}.$$

Hence A', $\bar{C}$ are the centres of perspective for the triads A, C.

(*b*) Again, let C^{**}, $\bar{C}^{**}$ be the centres of perspective of the two triads C, $\bar{C}$, then, still neglecting scalar factors,

$$C = \theta A = \bar{\theta} \bar{\theta} A = \bar{\theta} \bar{C} = \bar{\alpha} \Omega \bar{\alpha} \bar{C}$$

or
$$\alpha C = \Omega(\bar{\alpha} \bar{C}).$$

Hence the centres of perspective C^{**}, $\bar{C}^{**}$ are given by

$$\bar{C}^{**} = \bar{\theta} \bar{\alpha} \bar{C} = \bar{\theta} \bar{\alpha} \bar{\theta} A = \bar{\theta} \alpha \alpha \bar{\theta} A$$
$$= \bar{\alpha} I' \bar{\alpha} A = \alpha I' A,$$

or
$$\alpha \bar{C}^{**} = I' A,$$

and also
$$C^{**} = \bar{\theta} \bar{C}^{**} = \bar{\theta} \alpha I' A = \bar{\alpha} \Omega I' A = \bar{\alpha} I' A'$$

or
$$\alpha C^{**} = I' A'.$$

Hence A, A' are the centres of perspective for the triads C, $\bar{C}$, which completes the theorem.

We note also that, for example, the points a, a', c_1, $\bar{c}_1$ are collinear and form an equiharmonic range, and there will be nine such ranges.

7. *Triads on a conic.*

If the triads A, A' are in direct triple perspective then $A' = \psi A$ and by § 5, the centres of perspective C are given by $C = \psi^* A$, where

$$\psi^* = \begin{bmatrix} 1/p & 1/q & 1/r \\ 1/q & 1/r & 1/p \\ 1/r & 1/p & 1/q \end{bmatrix}.$$

The three points C_1, C_2, C_3 will be collinear provided $|\psi^*| = 0$, that is, provided

$$(1/p + 1/q + 1/r)(1/p + \omega/q + \omega^2/r)(1/p + \omega^2/q + \omega/r) = 0,$$

and the vanishing of any one of these factors implies that a, b, c, a', b', c' lie on a conic.

Taking the conic as $xy + yz + zx = 0$, any point on the conic may be written as $ta + t(t-1)b + (1-t)c$, where the points a, b, c correspond to values 1, ∞, 0 respectively of the parameter t. Let a', b', c' correspond to values λ, μ, ν of the parameter, so that

$$a' = \lambda a + \lambda(\lambda - 1)b + (1 - \lambda)c,$$
$$b' = \mu a + \mu(\mu - 1)b + (1 - \mu)c,$$
$$c' = \nu a + \nu(\nu - 1)b + (1 - \nu)c,$$

and the triads A, A' are then in direct triple perspective, provided

$$\lambda\mu(\mu-1)(1-\nu)=\lambda(\lambda-1)(1-\mu)\nu=(1-\lambda)\mu\nu(\nu-1),$$

or
$$\nu(\lambda-1)=\mu(\nu-1)=\lambda(\mu-1)=k, \text{ say.}$$

Eliminating λ, μ, we have

$$k(k+1)=(k+1)\nu(\nu-1)$$

whence
$$k=\nu(\nu-1),$$

leading to $\lambda=\mu=\nu$, which is degenerate, or, alternatively, $k=-1$. Hence the points a', b', c', a, b, c, in triple perspective, lie on a conic provided

$$\lambda(\mu-1)=\mu(\nu-1)=\nu(\lambda-1)=-1,$$

or
$$\mu=(\lambda-1)/\lambda, \quad \nu=1/(1-\lambda), \ \lambda\mu\nu=-1.$$

Inserting these values,

$$a'=\lambda a+\lambda(\lambda-1)b+(1-\lambda)c,$$
$$b'=\lambda(\lambda-1)a+(1-\lambda)b+\lambda c,$$
$$c'=(1-\lambda)a+\lambda b+\lambda(\lambda-1)c,$$

giving an ∞^1 triads, lying on the conic, in direct triple perspective with the points a, b, c. Forming the centres of perspective, $C=\psi^{*}A$, we have

$$C\ =(\lambda-1)a+b-\lambda c,$$
$$C\ =\qquad a-\lambda b+(\lambda-1)c,$$
$$C_3=-\lambda a+(\lambda-1)b-\lambda c,$$

whence it is clear that C_1, C_2, C_3 are collinear, since $C_1+C_2+C_3=0$ and that this line is $\alpha+\beta+\gamma=0$ which is the Hessian line of the three points a, b, c with respect to the conic. Hence it follows further that if the two triads A, A' are in triple perspective, and lie on a conic, they must have the same Hessian line with respect to the conic.

It is of interest to note that the cross ratios $\{abca'\}$, $\{bcab'\}$, $\{cabc'\}$ are equal so that given a, b, c a simple method of deriving the triads in triple perspective is established. In particular, if $\lambda=\frac{1}{2}$, 2 or -1 the cross ratios are harmonic and the corresponding points a, b, c, a', b', c' form a set in involution on the conic, and in this case

$$a'=a-\tfrac{1}{2}b+c, \quad b'=a+b-\tfrac{1}{2}c, \quad c'=-\tfrac{1}{2}a+b+c,$$

and the two triads a, b, c, a', b', c' are in quadruple perspective. More generally, for quadruple perspective $A'=\psi A$, where, in ψ, $p^3=q^3$, so that $p=q$, $p=\omega q$, or $p=\omega^2 q$. It can then be shown that the " fourth " centre of perspective is independent of the ratio $p:r$ and in fact gives the three points h, h_1, h_2 respectively of § 3, (6).

The results of the last section may be derived from the consideration of the two systems of generators on a quadric. We show first that if a (1, 1) correspondence is set up between the generators of the two systems, then the points of intersection of pairs of corresponding generators lie in a plane, and hence also the tangent planes to the quadric at these points are concurrent in a point.

Let A, G, B ; A', G', B' be six generators of a quadric, three belonging to one system and three to the opposite system. Let the points of intersection of AA', AB', BA', BB' be p, q, r, s respectively. Then if the generator G' passes through $r+\lambda s$, $p+\lambda q$ and the generator G passes through $p+\mu r$, $q+\mu s$, the generators G, G' intersect in the point

$$g=p+\mu r+\lambda q+\lambda\mu s\ ;$$

but if the generators G, G' are made to correspond, there is a (1, 1) correspondence between λ, μ, say

$$a + b\lambda + c\mu + \lambda\mu = 0.$$

Hence eliminating $\lambda\mu$, g is expressed in terms of three points and hence lies in a plane, π say. Further the tangent planes at g to the quadric must therefore meet in the pole of the plane π with respect to the quadric, and are therefore concurrent. To be more definite, let A, B, C be generators of the first system with parameters 1, ∞, 0, then any generator G of this system is

$$G = tA + t(t-1)B + (1-t)C,$$

and the two Hessian lines h_1, h_2 of A, B, C are given by $t = -\omega$, $t = -\omega^2$. Similarly, for the other system, let A', B', C' have parameters 1, ∞, 0 and so correspond to A, B, C; then

$$G' = tA' + t(t-1)B' + (1-t)C'$$

and again the two Hessian lines h_1, h_2 have parameters $-\omega$, $-\omega^2$. Generators G, G' intersect in the point $[GG'] = g$ which lies in the plane π. Let the traces of A, B, C, G, h_1, h_2 in the plane π be a, b, c, g, h_1, h_2, and the traces of A', B', C', G', h_1, h_2 in the plane π be a', b', c', g', h_1, h_2. Now the lines aa', bb', cc', h_1h_1, h_2h_2 are the traces in the plane π of the tangent planes at the points $[AA']$, $[BB']$, $[CC']$, $[h_1h_1]$, $[h_2h_2]$ and these tangent planes are concurrent, hence the joins aa', bb', cc' are concurrent at a point on the line of intersection of the tangent planes at $[h_1h_1]$, $[h_2h_2]$, that is, on the line L joining $[h_1h_2]$, and this line L is the Hessian line of the triads a, b, c and a', b', c'. Thus the triads a, b, c, a', b', c' have the same Hessian line and the first centre of perspective lies on L. Now permute A, B, C cyclically, leaving A', B', C' unaltered. This does not affect the Hessian lines or the line L. Hence, as above, ba', cb', ac' are concurrent in a point of L, and by a further permutation ca', ab', bc' are also concurrent in a point of L. Hence the three centres of perspective are collinear and lie on the common Hessian line of the two triads.

Further, the centres C_1, C_2, C_3 are the points of intersection, with the line L, of the tangent planes at $[A'A]$, $[A'B]$, $[A'C]$ which have parameters 0, ∞, 1 and hence C_1, C_2, C_3 with the points $[h_1h_2']$, $[h_1h_2']$ form equi-anharmonic ranges on the line L. W. R. A. & W. S.

The existence of sextuply perspective triangles can be shown from the well known figure of the nine inflexional points of a plane cubic curve.

The cubics $f \equiv s(x^3 + y^3 + z^3) + 6txyz = 0$ pass through nine fixed points. These cubics split up into a triad of lines if $s(s^3 + 8t^3) = 0$, that is, for four values of $s : t$. Thus twelve lines in sets of three pass through the nine points. In the diagram the nine points are labelled 1 to 9, and the four triangles are ABC, DEF, LMN, PQR. *Any two of these triangles are in sextuple perspective, with the other two as centres of perspectivity.* In fact, the 12 vertices of these triangles lie by fours on 9 straight lines, and thus form a figure dual to the original 9 points with 12 lines through them, 4 through each point.

Algebraically, it may be shown that any double point of $f = 0$ must lie on one of the loci $x^3 = y^3$, $y^3 = z^3$, $z^3 = x^3$, which are thus the 9 lines in question.

The result can also be proved by elementary synthetic methods. If in turn we project the range $456C$ on to AB from the points 1, 2, 3, we find $(798B) = (987B) = (879B)$, that is, these ranges are equi-anharmonic.

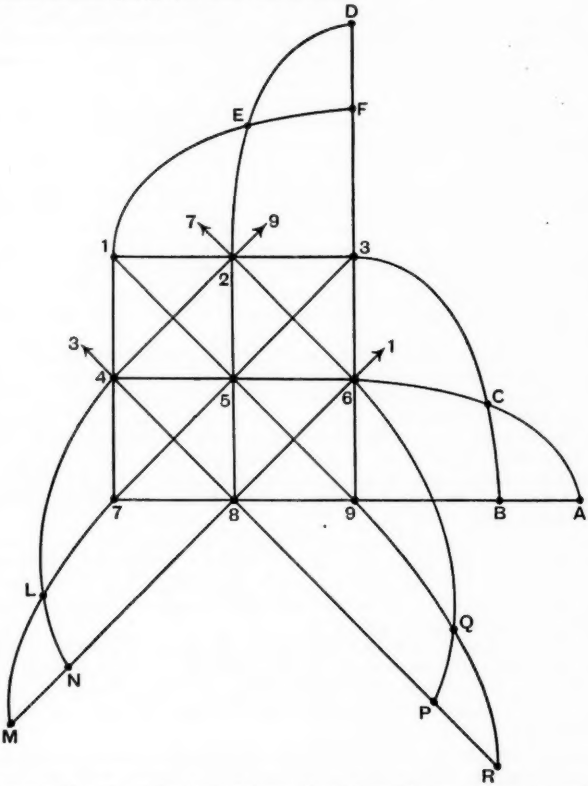
Also $7(456C) = (132C) = 4(789A)$. So $(789A) = (987B)$. Thus A and B are the Hessian points of 7, 8, 9 on the line.

Further $4(789A) = 1(798B)$. These pencils have the line 147 in common. Hence the intersections RNC are collinear. Similarly, from $5(789A) = 3(798B)$ DRC are collinear. Hence $CDRN$ are collinear.

Altogether we can thus find nine lines

$$\begin{array}{lll} ADPL, & BDQM, & CDRN \\ AEQN, & BERL, & CEPN \\ AFRM, & BFPN, & CFQL \end{array}$$

from which all the perspectivities are evident.



68 meets 59 at 1 : there is no intersection where the two appear to cross at the centre of the square 5698.

Conversely, if the system of sextuply perspective triangles is given, it follows that the twelve sides of the four triangles pass through nine points. For if ABC and DEF are in direct perspective from the centres L, M, N and in reverse perspective from P, Q, R, both P and L must lie on AD. Thus $ADPL$ is a straight line, and similarly all the lines listed above are straight. We thus have 12 points on 9 lines. The result to be proved is the dual of one already proved.

Thus the theory of sextuply perspective triangles is identical with that of the 9 points associated with the inflexions of a cubic. W. W. Sawyer.

THE CURVE OF PURSUIT.

BY C. C. PUCKETTE.

Find the path taken by a ship which pursues, and always directs its course towards, another ship (supposed moving in a straight line), their speeds being constant and in fixed ratio, and their initial directions at right angles.

The formulation and solution of this problem were first given by Pierre Bouguer* (1698–1758), a French hydrographer, and published by him in 1732. He called the path of the pursuing ship the *Courbe* (or *Ligne*) *de Poursuite*, and that of the one pursued the *Ligne de Fuite*. It has been argued, but with little force, that to Leonardo da Vinci (1510) is due the first real conception of a *curve* of pursuit, although *problems* of pursuit are met with from the time of Zeno's paradox of Achilles and the tortoise. In France the curve is also called the *Courbe de Chien*, and in Germany the *Verfolgungskurve* or *Hundekurve*, from its early association with the path taken by a dog in following its master.

Bouguer was Professor of Hydrography, first at Croisic in Lower Brittany, and then at Hâvre-de-Grace. His main work lay in the field of navigation, and it seems to have been certain aspects of the manoeuvring of ships at sea that caused him to investigate this curve. However, he makes it quite clear that the pursuing ship could catch its quarry much more quickly by "heading it off" than by merely following it, (assuming the line of flight remains a straight line).

After studying Bouguer's solution, P. L. M. de Maupertuis (1698–1759), discoverer of the principle of Least Action, also solved the problem, producing a much more concise analysis. He further states a generalisation (*Mémoires*, p. 16—see below) thus:

"Je vais chercher les courbes que doit décrire un Vaisseau pour en suivre un autre qui fuit par quelque courbe donnée que ce soit (en supposant aussi le rapport de vîtesses constant); et le Probleme se peut résoudre de la même maniere, lorsque le rapport des vîtesses seroit donné par quelque fonction des coordonnées des courbes."

His solution of this is left as a differential equation, into which are to be substituted the given functions. However, according to the Summary (see below), the substitution procedure did not work well, for it complicated even the straight line case, so that Maupertuis abandoned the method in favour of considering each case separately.

Both Bouguer's and Maupertuis' solutions, together with a Summary of the problem, appeared in the section on geometry in the *Histoire de l'Académie Royale des Sciences, Année M.DCCXXXII, Avec les Mémoires de Mathématique et de Physique, pour la même Année*† (one vol., qto., printed 1735). The details are:

1. Summary: *Sur les Courbes de Poursuites* (Author's name not given, but reference is made to Bouguer's and Maupertuis' papers, with these writers' views on certain points of the problem. A discussion follows on the kinds of curve obtained with special speed-ratios, leading to a definition of Algebraical or Geometrical curves in contrast to what are called "Mechanical" curves, (*Courbes Méchaniques*). The distinction is made between the two because the first type has an equation of finite degree, whereas the mechanical ones have

* Historical details taken mainly from *Enc. Britt.* (1947) and the histories of F. Cajori and D. E. Smith. Bouguer introduced the signs ⩾ and ⩽ (Cajori). He is best known for his measurement of a degree of the meridian near the equator, a work which occupied him some ten years.

† A copy of this volume may be seen in the Birmingham Central Library (their ref. 18820).

only a *differential* equation of finite degree.* For example, dx/x produces a "Mechanical" curve, dx/x^2 an Algebraical or Geometrical one). *Histoire*, p. 56 (5 pages).

2. Paper: *Sur de Nouvelles Courbes ausquelles on peut donner le nom de Lignes de Poursuites. Par M. Bouguer* (Read before the Academy on January 16, 1732). *Mémoires*, p. 1 (14 pages, plus one of diagrams). Reckoned by all authorities to be the first published account of the problem.

3. Paper: *Sur les Courbes de Poursuites. Par M. de Maupertuis. Mémoires*, p. 15 (2 pages, including a diagram).

Both (2) and (3) make free use of differentials (in Leibnitz's notation), giving a fair indication of the rapidity with which the technique spread on the continent. Of perhaps topical interest are the authors' explicit statements that the differential (of the independent variable) is a constant; we find the remarks "... faisant *ds* constant ..." and "... si l'on fait *dy* constant ...". One slight difference from modern notation may be noted: d^2y is written ddy, but powers are in general use.

Later British mathematicians, although possibly knowing of the existence of Bouguer's paper, do not seem to be aware of the extent of his results. In the Appendix to the *Gentleman's Diary* for 1839 appears a brief history† of the problem in this country, in which the comment is made that all previous [British] writers have not noticed that their solutions fail when the speed-ratio is unity [the logarithmic case]. Bouguer fully realised this, remarking that for this particular case, it was necessary to go back to the differential equation. He also derives some properties of the curve on differential and geometrical considerations, independently of the equation to the curve, and gives the formula‡ of equation (iii′) below. If all we require is to find where the ships finally meet (assuming they can), this formula is the simplest to obtain.

When the speed-ratio of pursuer to pursued is 2 : 1, Bouguer obtains a cubic equation of which he says (*Mémoires*, p. 8): "... on trouvera que la ligne de poursuite devient une des cinq paraboles divergentes de Monsieur Newton, sçavoir celle que ce fameux Géometre nomme *Noüée*, & qu'il regarde comme la 68^{me} du second genre. Cette courbe ... a deux branches parfaitement égales ... qui forment un folium ...".

Maupertuis' paper seems to have been largely overlooked,§ particularly his generalised problem. The *Gentleman's Diary* already mentioned gave a "general" solution, but only in so far as it removed the right angle condition at the start. Bouguer notes this wider approach, but chooses the rectangular one for simplicity, since no loss of principle is involved.

A special example of what we shall call Maupertuis' Problem, in which the line of flight was a circle (but with constant speed-ratio), proved rather difficult. It first appeared in 1859 (*Math. Monthly*, I, p. 249), but was not finally

* In Professor H. W. Turnbull's book, *Mathematical Discoveries of Newton*, there is (p. 26) a quotation from *De Analysi* which mentions "mechanical" curves. The *New English Dictionary* has the following note: "Applied to curves not expressible by equations of finite and rational algebraic form = transcendental. So called as admitting of production only by 'mechanical construction'". The final phrase is explained as "construction by the use of some apparatus as distinguished from 'tracing' by calculation of successive points."

† Reprinted in *Math. Gazette*, XV, No. 214, Note 1004.

‡ Also derived in *Math. Gazette*, XV, No. 208, p. 160: method essentially the same as Bouguer's.

§ Of the many sources consulted, only two refer to Maupertuis in this connection, namely, Peacock, *Calculus*, and *Amer. Math. Monthly*, (*loc. cit.*). Even Cajori and Smith fail to mention him.

solved until 1921 (*Amer. Math. Monthly*, XXVIII, pp. 54, 91, 278). The curve of pursuit was here called a *quadrifolium*. The problem of three dogs placed at the vertices of an equilateral triangle, and starting simultaneously with equal speeds, to chase one another, led to the *logarithmic spiral* as the curve of pursuit for each dog. (*Nouv. Corres. Math.*, III (1877), pp. 175, 280. This problem is generalised in *Johns Hopkins Univ. Circ.* (1908), p. 135.) I do not regard this latter problem as a particular case of Maupertuis'.

Present-day textbooks ignore the curve of pursuit altogether ; it seems to appear only in older books, for example, Tait and Steele, *Dynamics of a Particle* (p. 19), Thomson and Tait, *Natural Philosophy* (I, p. 28). The case considered is always that of Bouguer but using, as a rule, Maupertuis' method of solution.* One change, surprising for a maritime nation, is that we have abandoned ships at sea for more homely subjects : spider and fly, farmer and pig, etc.

No mention of three-dimensional curves of pursuit has been found.† Possibly this may prove to be a modern development, since one can reasonably expect some types of guided missiles to have curves of pursuit. Also, nothing essentially new appears to have been introduced since Maupertuis' problem, either here or abroad.

The problem considered below is similar to Bouguer's, but follows that of the *Gentleman's Diary* (*loc. cit.*) in removing the initial rectangular condition. The solution given is chiefly offered as an example of the two-dimensional (zero torsion) Frenet-Serret ‡ formulae in vector work. Mathematically we have to find the curve *whose arc is a constant multiple of the sum of the abscissa and subtangent.* Maupertuis gave this sum the name *resecte.* Our " tangent intercept " is not used here because of possible misunderstanding.

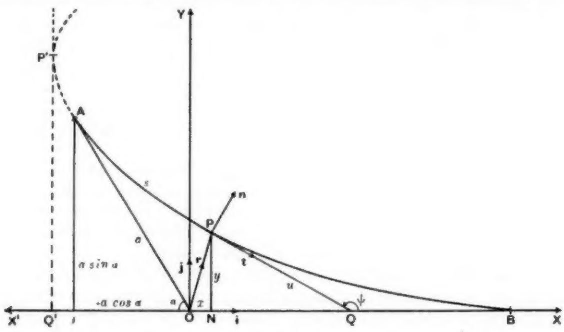


FIG. 1.

From the conditions of the problem, we have, on referring to Figs. 1 or 2, $AP = s$, $OQ = ms$; $\mathbf{i}$, $\mathbf{j}$, $\mathbf{t}$, $\mathbf{n}$ are unit vectors with their usual significance, $\overrightarrow{OP} = \mathbf{r}$, $\overrightarrow{OQ} = ms\mathbf{i}$, $\overrightarrow{PQ} = u\mathbf{t}$, where u is a scalar variable; $OA = a$, $\angle AOX' = \alpha$. In the sequel, κ is the curvature.

* Without acknowledgment, save in the cases cited. Bouguer is usually mentioned as the source of the problem.

† Except for C. Cailler, *Intro. Géom. à la Mécanique Rationnelle* (p. 508), who derives no particular solution, but provides a neat method of obtaining the general differential equation. (I am indebted to Professor Broadbent for this reference and for a copy of Cailler's method.)

‡ More correctly, Frenet formulae, first given by F. Frenet (1847).

By the addition of vectors, we have

$$ms\mathbf{i} = \mathbf{r} + u\mathbf{t}. \qquad \text{(i)}$$

Denoting derivatives with respect to s by a dash, we have

$$m\mathbf{i} = \mathbf{t}(1 + u') + u\kappa\mathbf{n}. \qquad \text{(ii)}$$

Multiplying (ii) by $\mathbf{t}\cdot$,

$$m\mathbf{t}\cdot\mathbf{i} = \mathbf{t}^2(1 + u') + u\kappa\mathbf{t}\cdot\mathbf{n},$$

that is,

$$-m\cos\psi\ (= mx') = 1 + u', \qquad \text{(ii}')$$

since $\mathbf{t}$ and $\mathbf{n}$ are perpendicular. Integrating, and noting that when

$$x = -a\cos\alpha, \quad u = a \quad \text{and} \quad s = 0,$$

gives

$$mx = u + s - a(1 + m\cos\alpha). \qquad \text{(iii)}$$

In the rectangular case, $\alpha = \frac{1}{2}\pi$, whence (iii) becomes

$$mx = u + s - a. \qquad \text{(iii}')$$

The curve touches the positive x-axis at B, where $u = 0$, $ms = x$ $(0 \leqslant m < 1)$. Substituting in (iii),

$$x_B = am(1 + m\cos\alpha)/(1 - m^2), \qquad \text{(iv)}$$

and for the rectangular case ($\alpha = \frac{1}{2}\pi$),

$$x_B = am/(1 - m^2). \qquad \text{(iv}')$$

To find y, multiply (ii) by $\mathbf{t}\wedge$, then

$$m\mathbf{t}\wedge\mathbf{i} = \mathbf{t}\wedge\mathbf{t}(1 + u') + u\kappa\mathbf{t}\wedge\mathbf{n},$$

that is,

$$m\sin\psi(= -my') = u\psi'.$$

But $y = u\sin\psi$, hence eliminating u,

$$-\frac{m\,dy}{y} = \frac{d\psi}{\sin\psi}.$$

Integrating, and noting that when $\psi = \pi - \alpha$, $y = a\sin\alpha$,

$$(ab/y)^m = \tan\tfrac{1}{2}\psi = \operatorname{cosec}\psi - \cot\psi \qquad \text{(v)}$$

$$= -\left(\frac{ds}{dy} + \frac{dx}{dy}\right), \qquad \text{(vi)}$$

where $b = \sin\alpha\ .\ (\cot\frac{1}{2}\alpha)^{1/m}$.

From (vi), $x + s$, x and s can each be found in terms of y, by the usual method of taking reciprocals. Further, on eliminating x and s, using (iii), we can find u in terms of y. Other sets of equations expressing u, s, x, y each in terms of ψ are also easily obtained. For reference purposes some of these are given below, distinguishing the cases $m \neq 1$ (algebraic curves) from $m = 1$ (logarithmic curves). For convenience, we have put $z = y/ab$, $c = a(1 + \cos\alpha)$,

$m \neq 1$.

$$2(1 - m^2)x = abz\{(1 - m)z^m - (1 + m)z^{-m}\} + 2am(1 + m\cos\alpha), \qquad \text{(vii)}$$

$$2(1 - m^2)s = 2a(1 + m\cos\alpha) - abz\{(1 - m)z^m + (1 + m)z^{-m}\}. \qquad \text{(viii)}$$

$m = 1$.

$$x = \tfrac{1}{2}c\log\{(a\sin\alpha)/y\} - (a^2\sin^2\alpha - y^2)/4c - a\cos\alpha, \qquad \text{(vii}')$$

$$s = \tfrac{1}{2}c\log\{(a\sin\alpha)/y\} + (a^2\sin^2\alpha - y^2)/4c. \qquad \text{(viii}')$$

All values of m.

$$u = \tfrac{1}{2}abz(z^m + z^{-m}) \quad \text{(ix)}$$

$$= ab\,(\cot \tfrac{1}{2}\psi)^{1/m} \,.\, \operatorname{cosec} \psi, \quad \text{(x)}$$

$$y = ab\,(\cot \tfrac{1}{2}\psi)^{1/m}. \quad \text{(xi)}$$

A further elimination gives the intrinsic equation.

As $m \to 1$, $x_B \to \infty$ for all $\alpha \neq \pi$. For $\alpha = \pi$,

$$x_B = am/(1+m) \to \tfrac{1}{2}a \quad \text{as } m \to 1,$$

as is otherwise obvious. When $m > 1$, u increases continually after

$$m \mid \cos \psi \mid > 1,$$

by (ii′). From this point we have the so-called *curve of flight**, since P and Q can then never meet. As $m \to 0$, $x_B \to 0$, so that P tends to "pounce" on Q before Q can get started. The x-axis is an asymptote when $m \geqslant 1$.

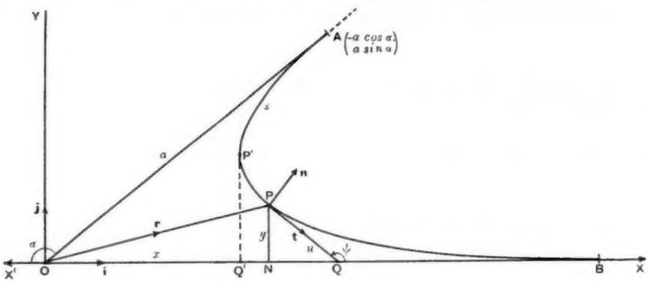


FIG. 2.

If A lies in the first quadrant so that $\tfrac{1}{2}\pi < \alpha < \pi$ (Fig. 2), u steadily decreases, for all m, until it is perpendicular to the x-axis at $P'Q'$. From here onwards we have the rectangular case, in which u increases or decreases depending on the values of m just discussed. When u is perpendicular to the x-axis, $\psi = \tfrac{1}{2}\pi$, and (v) gives $u = y = P'Q' = ab$. The rather surprising result† is therefore obtained that no matter how small a, $\pi - \alpha$ and m may be, provided none is zero, $b > 0$ and the advantage of initial position is not, of itself, sufficient to enable P and Q to meet. The cases are always subject to the values of m already discussed.

When u remains constant and equal to a, we no longer have Bouguer's problem, but a special case of Maupertuis'. The curve of pursuit becomes the well-known tractrix‡ (or tractory), whose intrinsic equation is $s = a \log \operatorname{cosec} \psi$ (in the rectangular case). The speed-ratio now varies with the position of P and Q, and is easily shown to be $-\cos \psi$, either by equations (i) and (ii) *et seq.*, or from the intrinsic equation. C. C. P.

* Tait and Steele, *loc. cit.*

† Noticed by Bouguer.

‡ See, for example, Edwards, *Diff. Calc.*, p. 357.

1753. *Principle and design of a new type of Stieltjes Integrator. . . .* Extensive tests with $h(x) = kx$ and $f(x) = (r^2 - x^2)^{\frac{1}{2}}$ indicate that accuracy of at least 0·1% can be expected when the curves are followed exactly.—From a review in *Mathematical Reviews*, Vol. 13, No. 7, p. 694. [Per Professor J. L. Burchnall.]

EQUATIONS IN POLYNOMIALS.

By Max Rumney.

The classical propositions of the elementary theory of numbers concerning divisibility, primes and linear indeterminate equations, are all based on the possibility of division with a remainder. That is, if a and b are two integers, b being positive, there exist unique integers q and r such that

$$a = qb + r, \quad 0 \leqslant r < b.$$

It is well known that there are similar theories for other systems in which the principle of division with a remainder is valid. One of the simplest examples is the system of *polynomials with rational coefficients*. If A and B are polynomials in an unknown x with rational coefficients, then there exist unique polynomials Q and R such that $A = QB + R$ and R is either 0 or has lower degree than B. The equation $A = QB + R$ is to be interpreted, of course, as an identity in x.

The classical theorem on linear indeterminate equations is that if a and b are two relatively prime integers the equation

$$au + bv = 1 \qquad \text{(1)}$$

is soluble in integers u, v. The practical process for finding the solution is to convert a/b into a continued fraction. Suppose, as we may without loss of generality, that a and b are positive, and let the continued fraction be

$$\frac{a}{b} = f_0 + \frac{1}{f_1 +}\,\frac{1}{f_2 +} \cdots \frac{1}{f_m}. \qquad \text{(2)}$$

From this continued fraction we can build up the convergents p_0/q_0, p_1/q_1, ... by the recurrence relations

$$p_n = f_n p_{n-1} + p_{n-2}, \quad q_n = f_n q_{n-1} + q_{n-2}, \qquad \text{(3)}$$

starting from $p_0 = f_0$, $q_0 = 1$, $p_1 = f_0 f_1 + 1$, $q_1 = f_1$. From the fact that a and b are relatively prime, it follows that $p_m = a$ and $q_m = b$. The relation

$$p_m q_{m-1} - p_{m-1} q_m = (-1)^{m-1} \qquad \text{(4)}$$

enables us to solve (1) by taking $u = q_{m-1}$ and $v = -p_{m-1}$, or $u = b - q_{m-1}$, $v = -a + p_{m-1}$ according as m is odd or even. We obtain a solution of (1) for which $0 < u < b$. There is only one solution satisfying this restriction, and it is easily seen that all other solutions are given by

$$u' = u + bt, \quad v' = v - at, \qquad \text{(5)}$$

where t is any integer.

A similar process applies to the equation

$$AU + BV = 1, \qquad \text{(6)}$$

where A and B are polynomials with rational coefficients, which have no polynomial factor; but there is an important difference. We can develop A/B in a continued fraction similar to (2), in which the partial quotients are polynomials with rational coefficients, and we can construct the convergents as before. But it no longer follows that $a = p_m$ and $b = q_m$; instead we have $A = hP_m$ and $B = hQ_m$, where h is a certain rational number. It is easily proved that h is the last remainder which occurs in the process of repeated division, before exact divisibility occurs. We now have

$$AQ_{m-1} - BP_{m-1} = \pm h, \qquad \text{(7)}$$

and so the polynomials Q_{m-1}/h and $-P_{m-1}/h$ (possibly with the signs changed) provide a solution of (6). In this solution, U is of lower degree than B and V

is of lower degree than A. The solution is unique under these restrictions; all other solutions are given as in (5) by

$$U' = U + BT, \quad V' = V - AT, \qquad \text{(8)}$$

where T is any polynomial with rational coefficients.

For example, let

$$A = \tfrac{14}{5}x^3 + \tfrac{137}{15}x^2 + \tfrac{59}{3}x + \tfrac{323}{15}, \quad B = \tfrac{14}{3}x^2 + \tfrac{41}{3}x + 16,$$

and we have to solve for U and V in $AU + BV = 1$. Developing A/B into a continued fraction as in (2), which for convenience we shall now write

$$(F_0, F_1, \dots, F_m),$$

we have

$$\frac{A}{B} = \left(\frac{3x+1}{5}, \frac{7x+3}{11}, \frac{2x+5}{3}\right),$$

from which we obtain

$$P_1 = \tfrac{1}{5}(3x+1), \quad Q_1 = 1;$$
$$P_2 = \tfrac{1}{55}(21x^2 + 16x + 58), \quad Q_2 = \tfrac{1}{11}(7x+3);$$
$$P_3 = \tfrac{1}{165}(42x^3 + 137x^2 + 295x + 323), \quad Q_3 = \tfrac{1}{33}(14x^2 + 41x + 48)$$

and $A = 11P_3$, $B = 11Q_3$, and $AQ_2 - BP_2 = -11$, whence

$$U = -Q_2/11 = -(7x+3)/121,$$
$$V = P_2/11 = (21x^2 + 16x + 58)/605,$$

and all other solutions are given by (8).

It is more interesting to consider polynomials with *integral coefficients*. Here the process of division with a remainder does not apply. Moreover, if A and B are polynomials with integral coefficients, having no common polynomial factor and no common numerical factor, it is not generally true that there are polynomials U and V with *integral* coefficients which satisfy (6). As we have seen, there are polynomials with *rational* coefficients which satisfy (6), and therefore there are some integers k for which the equation

$$AU + BV = k \qquad \text{(9)}$$

is soluble in polynomials U, V with integral coefficients. The *least positive integer k* with this property may be called the *arithmetical resultant* of A and B.

It is easily proved that any integer k for which (9) is soluble must be a multiple of the least such integer. In particular, it is known that the equation is soluble when k is the algebraic resultant of A and B, defined by Sylvester's determinant.* *Thus the arithmetical resultant is always a factor of the algebraic resultant.* For example, the algebraic resultant of $8x^2 + 6x + 7$ and $12x^2 + 7x + 6$ is

$$\begin{vmatrix} 8 & 6 & 7 & 0 \\ 0 & 8 & 6 & 7 \\ 12 & 7 & 6 & 0 \\ 0 & 12 & 7 & 6 \end{vmatrix} = 1088 = 64 \times 17,$$

while the arithmetical resultant is 17, since

$$(2x-3)(12x^2 + 7x + 6) - (3x-5)(8x^2 + 6x + 7) = 17.$$

The two are therefore not necessarily the same.

There is one simple case in which the arithmetical resultant is unity, and

* R. J. Walker, *Algebraic curves*, Theorem 9.6, W. V. D. Hodge and D. Pedoe, *Methods of algebraic geometry*, I, p. 151.

the equation (6) is soluble in polynomials U, V with integral coefficients. This is the case *when all the partial quotients in the continued fraction for A/B have integral coefficients*. For then $P_0, Q_0, P_1, Q_1, \ldots$ are also polynomials with integral coefficients, and it follows from $A/B = P_m/Q_m$ that $A = P_m$ and $B = Q_m$. For A, B have no common polynomial factor or numerical factor by hypothesis, and the same is true of P_m and Q_m since

$$P_m Q_{m-1} - P_{m-1} Q_m = \pm 1.$$

Take, for example,

$$A = 14x^5 - 47x^4 + 68x^3 - 60x^2 + 21x - 13,$$
$$B = 14x^3 - 47x^2 + 54x - 27.$$

Expressed as a continued fraction A/B, in our notation, is

$$(x^2 + 1, x - 1, 2x - 1, 7x - 13).$$

The penultimate convergent is

$$(2x^4 - 3x^3 + 4x^2 - x + 1)/(2x^2 - 3x - 2)$$

and thus

$$A(2x^2 - 3x - 2) - B(2x^4 - 3x^3 + 4x^2 - x + 1) = 1.$$

It may, however, still be possible to solve (6), even when $F_0, F_1, \ldots$ in $(F_0, F_1, \ldots, F_m)$ do not all have integral coefficients. For example, let

$$A = x^2 + a_1 x + a_2, \quad B = A' = 2x + a_1,$$

where A' is the derived function of A, then A/B as a continued fraction is

$$\left(\frac{2x + a_1}{4}, -\frac{8x + 4a_1}{a_1^2 - 4a_2}\right).$$

The algebraic resultant of A and B is

$$\begin{vmatrix} 1 & a_1 & a_2 \\ 2 & a_1 & 0 \\ 0 & 2 & a_1 \end{vmatrix} = a_1^2 - 4a_2,$$

and if we choose A so that its discriminant $a_1^2 - 4a_2 = 1$, the continued fraction becomes

$$(\tfrac{1}{4}(2x + a_1), -(8x + 4a_1)).$$

Now, since the algebraic resultant is unity, it must be possible to find U and V with integral coefficients, although the partial quotients have not integral coefficients. In this simple example, it is, nevertheless, clear, since the numerical factor 4 cancels out, that we have, in the final stages, partial quotients with integral coefficients.

If A and B are polynomials with no common polynomial factor, there exist polynomials U and V with rational (but not necessarily integral) coefficients such that

$$AU + BV = P. \qquad \text{......(10)}$$

One such pair is obtained by solving (6) and multiplying throughout by P.

Lemma I. If U_1, V_1 and U_2, V_2 are two solutions of

$$AU + BV = P,$$

then

$$U_1 V_2 - U_2 V_1 \equiv 0 \pmod{P}. \qquad \text{......(11)}$$

For we have

$$AU_1 + BV_1 = AU_2 + BV_2$$

or

$$A(U_1 - U_2) = B(V_2 - V_1).$$

Since A and B have no common polynomial factor it follows that

$$U_1 - U_2 = BT, \quad V_2 - V_1 = AT,$$

where T is some polynomial. Hence

$$\begin{aligned} U_1V_2 - U_2V_1 &= U_1(V_1 + AT) - V_1(U_1 - BT) \\ &= (AU_1 + BV_1)T = PT. \end{aligned}$$

That is, $\quad U_1V_2 - U_2V_1 \equiv 0 \pmod{P}$.

This identical congruence leads to an interesting theorem, which can be stated in a general form as follows :

Theorem I. If $U_1, V_1, \dots, U_{n+1}, V_{n+1}$ be any $n+1$ solutions of

$$AU + BV = P,$$

and $H_1(U, V), \dots, H_{n+1}(U, V)$ be any $n+1$ homogeneous polynomials of degree n in two unknowns, then the determinant

$$| H_i(U_j, V_j) | \quad i, j = 1, 2, \dots, n+1$$

is a *polynomial* in x and

$$| H_i(U_j, V_j) | \equiv 0 \pmod{P^{n(n+1)/2}}.$$

For, by the rule for multiplying two determinants, the above determinant is equal to a certain constant determinant multiplied by

$$\begin{vmatrix} U_1^n & U_2^n & \dots & U_{n+1}^n \\ U_1^{n-1}V_1 & U_2^{n-1}V_2 & \dots & U_{n+1}^{n-1}V_{n+1} \\ \dots & \dots & \dots & \dots \\ V_1^n & V_2^n & \dots & V_{n+1}^n \end{vmatrix}.$$

This factorises as the product of the $\frac{1}{2}n(n+1)$ determinants $U_jV_k - U_kV_j$, where $j < k$, and each such determinant is a multiple of P. That is,

$$| H_i(U_j, V_j) | \equiv 0 \pmod{P^{n(n+1)/2}}.$$

We shall end with an arithmetical example to illustrate this theorem. Let u_1, v_1 ; u_2, v_2 ; u_3, v_3 be any three solutions of $au + bv = c$, and $h_1(u, v)$, $h_2(u, v)$, $h_3(u, v)$ three homogeneous quadratics in the two unknowns u and v. For example,

$$3u - 7v = 5 : \ u_1 = 11, v_1 = 4 \ ; \ u_2 = 18, v_2 = 7 \ ; \ u_3 = -3, v_3 = -2.$$
$$h_1 = u^2 + uv + v^2, \ h_2 = u^2 - uv + v^2, \ h_3 = 2u^2 - uv - 2v^2.$$

Then

$$\begin{vmatrix} h_1(u_1, v_1) & h_1(u_2, v_2) & h_1(u_3, v_3) \\ h_2(u_1, v_1) & h_2(u_2, v_2) & h_2(u_3, v_3) \\ h_3(u_1, v_1) & h_3(u_2, v_2) & h_3(u_3, v_3) \end{vmatrix} \equiv 0 \pmod{c^3}.$$

In fact,

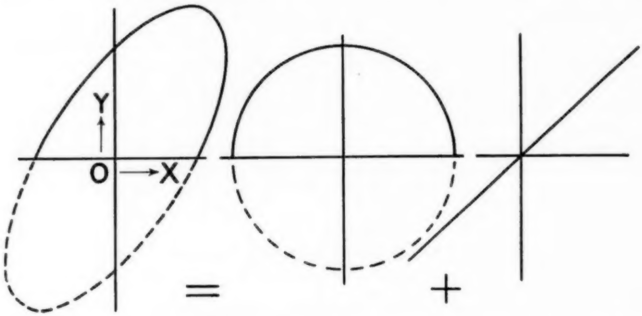
$$\begin{vmatrix} 181 & 399 & 19 \\ 93 & 247 & 7 \\ 166 & 424 & 4 \end{vmatrix} = 73000 = 584 \times 5^3 \equiv 0 \pmod{5^3}.$$

My grateful acknowledgments are due to Professor H. Davenport for his valuable advice on the preparation of this article. M. R.

MATHEMATICAL NOTES

2358. *On a result in the theory of plane curves.*

During the course of an investigation in two dimensional potential theory it became necessary to express certain boundaries as the sum of two components. One of these had symmetry about an axis in its plane and the other anti-symmetry. Since the result, although elementary in derivation, is completely general and does not appear to be well known—at least in explicit form—it was thought worth while to compose this short note on the process.



Theorem. Given *any* plane curve, not necessarily continuous, and *any* pair of rectangular axes in the plane of that curve. Then there exists a unique resolution of that curve into two curves, one symmetric and the other antisymmetric, about either of the axes. For, suppose the curve to be given by the relation :

$$y = f(x)$$

about the given axes ($f(x)$ is not necessarily a continuous function but can be, for example, a point set), and let the symmetric and antisymmetric components about the y axis be denoted by :

$$y = S(x) \quad \text{and} \quad y = A(x)$$

respectively.

Then we have :

(*a*) from the assumed symmetry :

$$S(-x) = S(x), \quad A(-x) = -A(x),$$

(*b*) $$f(x) = S(x) + A(x)$$

whence $$f(-x) = S(-x) + A(-x)$$
$$= S(x) - A(x) \quad \text{from } (a).$$

Thus, adding and subtracting,

$$S(x) = \tfrac{1}{2}[f(x) + f(-x)]$$
$$A(x) = \tfrac{1}{2}[f(x) - f(-x)]$$

which establishes the result.

As an example of the application of the theorem to a point set may be mentioned the well-known result :

" Any matrix can be expressed as the sum of a symmetric matrix and an antisymmetric matrix ".

Another result is :

" Any definite integral can be reduced to an integral between limits 0 and ∞."

Finally, in the field of continuous curves, there is the interesting result :

" Any centric conic, referred to rectangular axes through its centre, can be expressed as the sum of a symmetrically placed central conic, of the same sort, and of a line segment."

A resolution of this kind is shown in the figure, the proof we leave to the reader. A. D. BOOTH.

2359. *The binomial theorem for any index.*

The following proof is possibly a little shorter than several existing ones: it uses the theorem on differentiation of a power series.

Let $$m_r = m(m-1)\ldots(m-r+1)/r!,\ m_0 = 1.$$
Then $$(m+1)_r - m_r = m_{r-1} \quad (r \geqslant 1). \qquad \ldots\ldots(1)$$
If
$$f(m) = \sum_0^\infty m_r x^r \quad (|x| < 1), \qquad \ldots\ldots(2)$$
then
$$f(m+1) - f(m) = \sum_1^\infty \{(m+1)_r - m_r\} x^r$$
$$= x \sum_1^\infty m_{r-1} x^{r-1}, \quad \text{by (1)}$$
$$= x f(m), \quad \text{by (2)}.$$
Hence
$$\frac{f(m+1)}{(1+x)^{m+1}} = \frac{f(m)}{(1+x)^m} = \frac{f(m-1)}{(1+x)^{m-1}} = \ldots .$$
Thus $$f(m) = \phi(x)(1+x)^m, \qquad \ldots\ldots(3)$$
where $\phi(x)$ is independent of m. But
$$df(m)/dx = \sum_1^\infty m_r r x^{r-1} = m \sum_0^\infty (m-1)_r x^r$$
$$= m f(m-1), \quad \text{by (2)}. \qquad \ldots\ldots(4)$$
That is, by (3) and (4),
$$(1+x)^m \phi'(x) + m(1+x)^{m-1}\phi(x) = m(1+x)^{m-1}\phi(x)$$
and hence
$$\phi'(x) = 0, \quad \phi(x) = K, \text{ a constant.}$$
Thus $$f(m) = K(1+x)^m$$
and $$K = f(0) = 1;$$
that is, $$f(m) = (1+x)^m.$$

A. J. CARR.

2360. *Note on reverse numbers.*

In connection with Notes 2958 and 2065 on the reversion of numbers, we have an analogous property for the sum of two such numbers if we write the properties in their direct forms, namely

$$\overline{ab} - \overline{ba} = (a-b)(A^k - 1)$$
$$\overline{ab}^2 - \overline{ba} = (a^2 - b^2)(A^{2k} - 1)$$

where k is the number of digits in each group a and b and A is the number base. Then the corresponding property is

$$\overline{ab} + \overline{ba} = (a+b)(A^k + 1).$$

For example,

$$32 + 23 = 5 \,.\, 11,$$

$$2137 + 3721 = 58 \,.\, 101.$$

A. BUCKLEY.

2361 *Rate of work problems and the reciprocal nomogram.*

With reference to Notes 1558, 2110 and 2108, since writing Note 2110 I have found that it is easy to show, by coordinate geometry that the method of Notes 1558 and 2110 can be extended to a perfectly general reciprocal nomogram.

There are thus simple straight line reciprocal nomograms. As far as I can see, each is capable of dealing easily with a few problems that the other will just not tackle at all. The rate of work form will deal with two or more men, one of whom starts after the other. It also gives easily the proportion of the work done by each. The method implied in Fig. 4 of Note 1558 is also capable of being put to a greater variety of uses than I first realised. The three-coordinate form will deal with some optical problems that the " rate of work " will not deal with, though the " rate of work " method will do some optical problems.

I should be most grateful to anyone who could give me demonstrations of any nomograms in simple language like my Note 1558.

C. DUDLEY LANGFORD.

2362. *The point of contact of the nine-points circle and incircle.*

In attempting to demonstrate for many years Feuerbach's theorem by diagram, I have repeatedly found that the nine-points circle either fails to meet the incircle or, worse still, touches it more than once! I was recently tempted to investigate the position of the point of contact, so that the following analysis of the problem may be of interest to others ; the conclusion is that the point of contact lies on that segment of the incircle facing the mean angle of the triangle.

Let the incircle touch the sides at X, Y, Z and the nine-points circle at P_1. Using areal coordinates, the equation of the incircle is

$$a^2yz + b^2zx + c^2xy - (x+y+z)\{(s-a)^2x + (s-b)^2y + (s-c)^2z\} = 0$$

and that of the nine-points circle is

$$a^2yz + b^2zx + c^2xy - (x+y+z)\{\tfrac{1}{2}bc \cos A \,.\, x + \tfrac{1}{2}ca \cos B \,.\, y + \tfrac{1}{2}ab \cos C \,.\, z\} = 0.$$

Hence the common tangent, which is the radical axis, has the equation

$$\Sigma\{(s-a)^2 - \tfrac{1}{2}bc \cos A\}x = 0,$$

that is,

$$\Sigma\{(b+c-a)^2 - (b^2+c^2-a^2)\}x = 0$$

or

$$\Sigma(a-b)(a-c)x = 0. \qquad \text{(i)}$$

Now the equation of the incircle may be written

$$\Sigma\sqrt{\{(s-a)x\}} = 0,$$

and the tangent at P_1 is

$$\Sigma x\sqrt{\{(s-a)/x_1\}} = 0.$$

But this is the same line as (i) above, and hence

$$(b-c)\sqrt{\{(s-a)/x_1\}} = (c-a)\sqrt{\{(s-b)/y_1\}} = (a-b)\sqrt{\{(s-c)/z_1\}}$$

so that P_1 is the point

$$(b-c)^2(s-a), \quad (c-a)^2(s-b), \quad (a-b)^2(s-c).$$

Now, Y is the point $(s-c, 0, s-a)$ and Z is $(s-b, s-a, 0)$. Thus the equation of the line YZ is

$$-(s-a)x+(s-b)y+(s-c)z=0.$$

Substitute the coordinates of $A(1, 0, 0)$ in the left-hand side of this equation and we have $-(s-a)$, which is negative. Now substitute the coordinates of P_1 in this left-hand side* and we have

$$-(s-a)^2(b-c)^2+(s-b)^2(c-a)^2+(s-c)^2(a-b)^2,$$

so that P_1 and A lie on the same side of YZ provided this expression is also negative, that is, if

$$\begin{aligned}(s-a)^2(b-c)^2 &> (s-b)^2(c-a)^2+(s-c)^2(a-b)^2\\ &=\{(s-b)(c-a)+(s-c)(a-b)\}^2-2(s-b)(s-c)(c-a)(a-b)\\ &=\{s(c-b)+a(b-c)\}^2-2(s-b)(s-c)(c-a)(a-b)\\ &=(s-a)^2(b-c)^2-2(s-b)(s-c)(c-a)(a-b),\end{aligned}$$

that is, provided

$$(s-b)(s-c)(c-a)(a-b)>0,$$

that is, provided

$$(c-a)(a-b)>0,$$

that is, provided

$$c>a>b \quad \text{or} \quad c<a<b,$$

that is, provided a is the mean side of the triangle.

Thus the point of contact of the nine-points circle and incircle is on that arc of the incircle which faces the mean angle. F. M. GOLDNER.

2363. *Motion of a rocket.*

The problem of determining the motion of a rocket is usually solved by a method involving the use of infinitesimals. The method, well known, is to write down the momentum of the rocket at time t and equate this to its momentum at time $t+\delta t$ together with the momentum of the additional burnt gases expelled in the interval δt.

The following method is an alternative approach based on the construction of a simple integral equation. Let

M_0 = initial mass of rocket at time $t=0$,
$m(t)$ = mass burnt per unit time at time t,
$V(t)$ = efflux velocity of gases (relative to the rocket) at time t,
$v(t)$ = velocity of rocket at time t.

The mass of the rocket at time t is

$$M_0-\int_0^t m\,dt,$$

and its momentum is then

$$\left\{M_0-\int_0^t m\,dt\right\} v.$$

Now at time t, the momentum of the expelled gases (measured in the direction

* Although the coordinates used for P_1 are not the absolute ones, they are all positive and hence may be used for determining on which side of a given line the point lies.

of motion of the rocket) increases at the rate $m(v-V)$. Hence the momentum of all the burnt gas at time t is

$$\int_0^t m(v-V)\,dt.$$

For a rocket moving horizontally, the total momentum of the entire system is constant at any time t. Hence

$$\left\{M_0-\int_0^t m\,dt\right\}v+\int_0^t m(v-V)\,dt=\text{constant}.$$

Differentiating this equation, we obtain

$$-mv+\left\{M_0-\int_0^t m\,dt\right\}\frac{dv}{dt}+m(v-V)=0,$$

so that

$$\frac{dv}{dt}=mV\Big/\left\{M_0-\int_0^t m\,dt\right\}.$$

F. Chorlton.

2364. *On Note* 2075 : *Nearly-isosceles right-angled triangles.*

With reference to the note by B.D.P., the following method of finding the values of the numerators and denominators of the odd convergents is useful.

The numerators 1, 7, 41, 239, etc., have a Scale of Relation $1-6x+x^2$. The exact values are given by :

$$P_n=\tfrac{1}{2}\left[\frac{\surd 2+1}{(3-2\surd 2)^{n-1}}-\frac{\surd 2-1}{(3+2\surd 2)^{n-1}}\right].$$

For positive integral values of n, the second term may be disregarded, so leaving the value of P_n :

$$P_n=1{\cdot}207107.5{\cdot}828457^{(n-1)}$$

to nearest whole number.

Hence, to nearest whole number :

$$\log P_n={\cdot}7655\,(n-1)+{\cdot}0817.$$

Similarly, the denominators may be found by :

$$Q_n={\cdot}8535533.5{\cdot}828457^{(n-1)}$$

to nearest whole number.

Or, to nearest whole number :

$$\log Q_n={\cdot}7655\,(n-1)+{\cdot}0688.$$

For higher values, 4-figures logarithms and even 7-figure tables will not suffice. However, for quite a considerable range of values of n greater than $n=5$, the correct values of P_n and Q_n can be found from the connection (to the nearest whole number) :

$$P_{(n+1)}=5{\cdot}828457P_n,$$
$$Q_{(n+1)}=5{\cdot}828457Q_n.$$

James A. H. Hunter.

2365. *An old result in a new dress.*

The series

$$\tfrac{1}{1}+\tfrac{1}{2}+\tfrac{1}{3}+\ldots+\tfrac{1}{9}+\tfrac{1}{10}+\tfrac{1}{11}+\ldots$$

is divergent.

But if the denominators of the terms are read as numbers in scale eleven or any higher scale, the series is convergent, and the sum is greater than 2·828 and less than 26·29. The convergence is rather slow. I estimate that, to find the last number by direct addition, one would have to work out 10^{90} terms, to about 93 places of decimals. G. W. Brewster.

2366. *The sundial.*

Fig. 68 and the description of the sundial (horizontal type) on p. 55 of the *Trigonometry Report* seem to me, and to many others, to be hard to follow.

The following description was evolved over 20 years ago when I was teaching geography specialists. It explains a method for making the horizontal type, the south wall type, and can be easily extended for a wall that does not face due south. If the reader will draw his own figure, line by line, as he reads the description, he will find it easier to understand the figure drawn here.

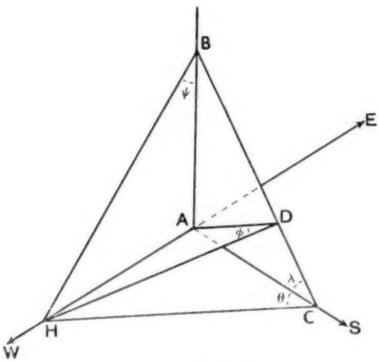


Fig. 1.

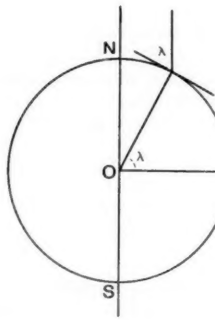


Fig. 2.

In Fig. 1, AB is a vertical line on a wall which faces due south, AC is a line on the ground at right angles to the wall, BC is a thin rod (the style or gnomon) parallel to the earth's axis, so that angle ACB is λ. AD is drawn perpendicular to BC, EW is the line in which the wall and ground meet. The plane through D and EW is at right angles to the thin rod and so parallel to the plane of the equator.

The shadow of the rod on the plane DEW (whichever side of the plane the sun may be) rotates about D at the rate of 15° per hour. If the angle ϕ is 15°, DH shows the shadow at 11 a.m., and H and C are points on the shadow of the rod. Hence CH is the shadow of the rod on the ground. Similarly, BH is the shadow on the south-facing wall.

It is easy to draw the figures in the separate planes of Fig. 1 and so to graduate the horizontal and south wall dials. Draw the figures of the planes in the following order: ABC, DEW (making the angle ϕ 15°, 30°, 45°, etc.), then the ground dial and the south-wall dial.

To find the connection between θ, ϕ, ψ and the latitude λ we have

$$\tan\theta = AH/AC, \quad \tan\phi = AH/AD, \quad \sin\lambda = AD/AC.$$

Hence $\tan\theta = \tan\phi \sin\lambda$;

similarly, $\tan\psi = \tan\phi \cos\lambda$.

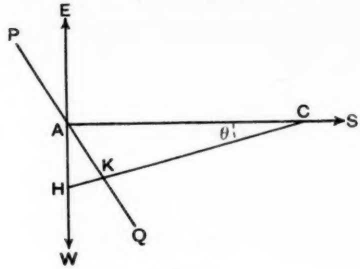


FIG. 3.

If the wall does not face due south, use Fig. 3. It shows the ground plan of Fig. 1; suppose that PQ is the line in which the wall and the ground meet. Then the line joining K to the point B will give the shadow on the wall.

A. W. SIDDONS.

2367. *The ratio of two quadratics.*

1. Note 2049 (A. A. Krishnaswami Ayyangar, *Math. Gazette*, XXXIII, 123–125) deals with the question of constructing two quadratics whose ratio shall have given rational maximum and minimum values. The following method may be of interest.

Case I. The denominator is to have no real zeros and the ratio varies between two given values L and U. Let

$$\alpha = \tfrac{1}{2}(U + L), \quad \beta = \tfrac{1}{2}(U - L) > 0.$$

Then the function

$$y = \alpha + \beta \cos(\theta - \gamma) = \alpha + p\cos\theta + q\sin\theta,$$

where $p^2 + q^2 = \beta^2$, assumes values between $(\alpha \pm \beta)$ inclusive, *i.e.* between U and L. Now, put $\tan\frac{1}{2}\theta = t$, obtaining

$$y = \{(\alpha + p) + 2qt + (\alpha - p)t^2\}/(1 + t^2), \qquad (1)$$

as a function of t with the required properties. The denominator may then be disguised by replacing t by a linear function of x with rational coefficients. The method is clearly reversible and can be used to deal with a rational function of x, whose denominator is a quadratic without real linear factors. This remark also applies, *mutatis mutandis*, to the cases dealt with below.

In (1), p and q are any rational numbers for which $p^2 + q^2 = \beta^2$. The procedure is simplest if we take one of p, q zero. For example, if $L = 5$, $U = 9$, then $\alpha = 7$, $\beta = 2$. Taking $p = 0$, $q = \beta$, so that $\gamma = -\frac{1}{2}\pi$,

$$y = 7 + 2\sin\theta = (7 + 4t + 7t^2)/(1 + t^2).$$

Now put $t = x - \frac{1}{2}$, to obtain $y = (27 - 12x + 27x^2)/(5 - 4x + 4x^2)$, one of the examples used by Mr. Ayyangar. On the other hand, $q = 0$, $p = \beta$ (so $\gamma = 0$) and the same equation for x would give

$$y = 7 + 2\cos\theta = (41 - 20x + 20x^2)/(5 - 4x + 4x^2).$$

2. *Case* II. The denominator is to have real zeros so that the graph of the function has real asymptotes and is in three pieces.

The denominator $(1 + t^2)$ in (1) is clearly to be replaced by $(1 - t^2)$, so that hyperbolic functions are indicated. We have two sub-cases, according to

whether we start with hyperbolic cosines or sines; this is to be contrasted with Case I, where sines and cosines are on the same footing.

Case II*a*. The function $y=\alpha+\beta\cosh(z+\delta)$ has just one turning value $(\alpha+\beta)$, which is a minimum when β is positive and a maximum when β is negative. Expanding and putting $p=\beta\cosh\delta$, $q=\beta\sinh\delta$, so that $p^2-q^2=\beta^2$ and p has the same sign as β, we have

$$y=\alpha+p\cosh z+q\sinh z.$$

Then, putting $t=\tanh\frac{1}{2}z$, we conclude that if $|t|<1$, $p^2-q^2=\beta^2$, then

$$y=\{(p+\alpha)+2qt+(p-\alpha)t^2\}/(1-t^2) \quad\ldots\ldots\ldots\ldots(2)$$

has a minimum $(\alpha+\beta)$, when $p>0$, and a maximum $(\alpha-\beta)$, when $p<0$.

The lines $t=\pm1$ are asymptotes of the graph of the function in (2), so that $|t|<1$ gives the region between these asymptotes. To obtain the region outside, re-write (2) as

$$y=\alpha-p\frac{t^2+1}{t^2-1}-q\frac{2t}{t^2-1},$$

and put $t=\cosh\frac{1}{2}z'$, obtaining

$$y=\alpha-p\cosh z'-q\sinh z'=\alpha-\beta\cosh(z'+\delta),$$

with β and δ as before (but z' differs from z). From this we see that for $|t|>1$, the function (2) has a maximum $(\alpha-\beta)$ when $p>0$, and minimum $(\alpha+\beta)$, when $p<0$.

Combining the two results: the function defined by (2), with $p^2>q^2$, has two turning values $(\alpha\pm\beta)$, where $\beta^2=p^2-q^2$; one turning value is a maximum, the other a minimum. When p is positive, the minimum is between the asymptotes $t=\pm1$; and when p is negative, the maximum is between the asymptotes. The difference between this case and case I, considered as answers to our original question, lies in the different relations between p, q and β, and in the fact that (2) is (1) with the sign of t^2 changed.

Given turning values U and L, we define α, β as in Case I and choose p, q to be rational, $p^2-q^2=\beta^2$. The simplest case is $q=0$, the case $p=0$ being inadmissible. For example, taking U, L and hence α, β as in the example of case I, and taking $p=\beta=2$, we have

$$y=\alpha+\beta\cosh z=7+2\frac{1+t^2}{1-t^2}=(9-5t^2)/(1-t^2).$$

The substitution $t=x-\frac{1}{2}$ then gives $y=(31+20x-20x^2)/(3+4x-4x^2)$ as a rational function with turning values $y=5$, $y=9$.

3. *Case* II*b*. The function $y=\alpha+\beta\sinh(z+\epsilon)$ has no turning values, so that when it is expressed in terms of t we get a rational function with no turning points. This function will be given by (2) but is distinguished from one belonging to Case II*a* by the condition $p^2<q^2$ (since $p=\beta\sinh t$, $q=\beta\cosh t$). The simplest case will be $p=0$. For example,

$$y=7+2\sinh z=(7+4t+7t^2)/(1-t^2).$$

4. It is of interest to notice that the above shows us that the graph of y against t is the result of subjecting graphs of trigonometric and hyperbolic functions to fairly simple transformations of the abscissae. Certain of the main features of the original graphs are still present in the transformed graphs. In Case I we transform one cycle of a cosine graph and the shape is by no means completely lost. Similarly, in Case II*a*, the part of the graph between the two parallel asymptotes still preserves the U-shape of the catenary from which it is derived; the same is seen to be true of the other parts of the graph if we

imagine them joined up at infinity. In this case we are transforming the two catenaries

$$y = \alpha \pm \beta \cosh (z + \delta),$$

one of which is inverted. Case IIb is similar.

5. It may be remarked that, in Case II, a function whose graph has given asymptotes and rational, but unspecified, turning values is most easily obtained as follows :

$$y = c + \frac{A}{x-a} + \frac{B}{x-b}$$

gives asymptotes $x = a$, $x = b$ and $y = c$, and its turning points are given by

$$A(x-b)^2 + B(x-a)^2 = 0.$$

The turning points then have rational x, and hence rational y, if $\sqrt{(-A/B)}$ is rational, assuming a, b and c to be rational. On the other hand, there are no turning points if A/B is positive.

Finally, the student (for whom I suppose this is all intended!) might find it interesting to relate the graphs of the following degenerate cases to those of the hyperbolic functions :

(i) the hyperbola

$$y = c + \frac{2A}{x-a},$$

obtained from

$$y = c + \frac{A}{x-a} + \frac{A}{x-b},$$

by the limiting process $b \to a$;

(ii) $$y = c + \frac{C}{(x-a)^2}$$

obtained from

$$y = c + \frac{A}{x-a} - \frac{A}{x-b},$$

by the limiting process $b \to a$, $A(a-b) \to C$.

R. D. Lord.

2368. *A new quadrature of the circle.*

$$\int_0^\pi x f(\sin x)\,dx = \int_0^\pi (\pi - x) f(\sin \overline{\pi - x})\,dx$$
$$= \int_0^\pi (\pi - x) f(\sin x)\,dx,$$

so that

$$2\int_0^\pi x f(\sin x)\,dx = \pi \int_0^\pi f(\sin x)\,dx.$$

Now, suppose that $f(\alpha) = \alpha \sin^{-1} \alpha$, so that $f(\sin x) = x \sin x$. Then

$$2\int_0^\pi x^2 \sin x\,dx = \pi \int_0^\pi x \sin x\,dx,$$

or $$2(\pi^2 - 4) = \pi^2,$$

whence $$\pi = 2\sqrt{2}.$$

George Tyson.

2369. *The prime lattice.*

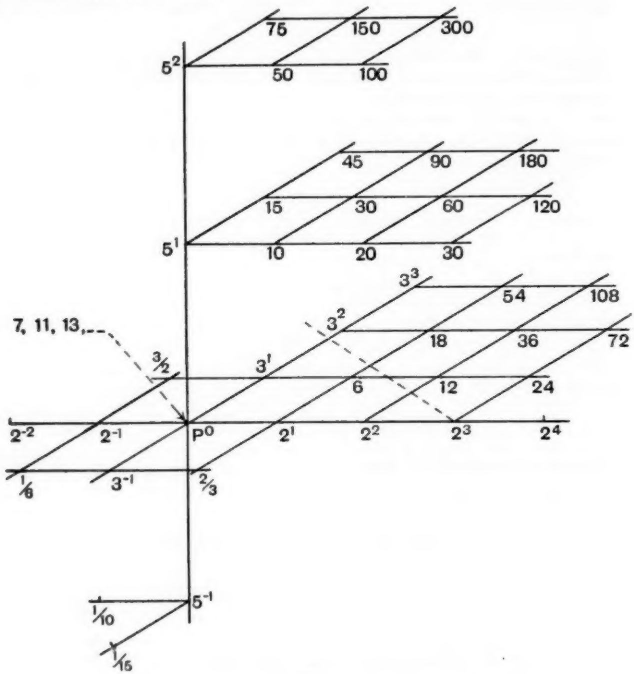
Besides having many properties interesting on their own account, the Prime Lattice suggests a fresh approach to certain problems of Number Theory.

Construction.

Consider a cubic lattice in n-space.

Associate the lattice point x units from the origin along the ith axis ($i=1$, $2, \ldots$) with p_i^x, where p_i is the ith prime. x is integral but not necessarily positive.

Each lattice point, Q, not on the axes is then associated with the product of all numbers occurring at the projections of Q on to the various axes. Thus, in the plane defined by the axes along which powers of the primes 3 and 5 are represented, the lattice point with co-ordinates (2, 1) would be associated with the number $3^2 \cdot 5^1 = 45$.



FIG

The prime lattice: the contour $C=8$ shown in the (2, 3) plane.

Elementary properties.

(1) All integers occur as lattice points in the principal n-octant.

(2) There is a 1 : 1 correspondence between all lattice points and the rational numbers. The distinction between a lattice point and its associated number will henceforth be ignored.

(3) All primes—and their reciprocals—occur at the intersection of the axes with the unit n-sphere.

(4) All lattice points lying on a straight line, not necessarily parallel to any axis, are in *GP*.

Connected with this important property we have :

(*a*) Sets of lattice points lying on lines parallel to one another are in *GP*'s of the same ratio.

(*b*) The origin bisects the straight line joining reciprocals.

(*c*) If a straight line be drawn from the origin to any number, Q, and then projected for n times its length beyond Q, it will terminate at the point associated with Q^n.

(*d*) On a sphere of any radius constructed about any number, Q, as centre, the product of a pair of lattice points diametrically opposite to one another will be Q^2.

(5) Square numbers (in general, numbers which are perfect nth powers) will lie on a sub-lattice of edge equal to 2 units (in general, of edge equal to n units).

(6) If $a+b=c$ and the points associated with a, b and c form a triangle, then any other congruent triangle ABC with $AB \parallel ab$, $BC \parallel bc$ and $AC \parallel ac$ will be such that $A+B=C$.

(7) All quadratfrei numbers are points on the unit cube in the principal n-octant.

(8) The number of divisors, $d(n)$, of a number equals the number of lattice points in the rectangular block formed by the projections of n on to the axes.

(9) The total number of prime factors of a number, $\Omega(n)$ equals the length of the shortest route from n to the origin without leaving the lattice.

(10) The number of different prime factors of a number, $\omega(n)$ equals the dimensionality of the lattice necessary to represent n.

(11) The contour of constant value in any plane of the lattice is a straight line. In the plane defined by the axes along which powers of p_i and p_j are represented, its gradient will be $-\log p_i/\log p_j$. More generally, the surface of constant value is a hyper-plane which intersects the ith axis at a distance of $\log C/\log p_i$ where C is the constant value in question.

(12) Lines normal to the constant value contour represent *GP's* of the maximum ratio per unit distance obtainable in the n dimensions of the lattice.

(13) The logarithm of a number occurring on the axis along which powers of p_i are represented, will be equal to the distance of the number from the origin multiplied by $\log p_i$. More generally, the logarithm of any number is equal to its distance from the origin divided by the length of the intercept made by the contour of value e on the line joining the origin to the number in question.

(14) The common factors of the numbers Q, R, S... will all lie in the rectangular block common to the blocks formed by the projections of Q, R, S, ... on to the axes. The H.C.F. of Q, R, S, ... will be the maximum number represented in the common block.

(15) The L.C.M. may be similarly defined.

Problems.

(1) If $n=2^a3^b5^c\ldots$ show, by consideration of the lattice that

$$d(n)=(a+1)(b+1)(c+1)\ldots \quad \text{and} \quad \Omega(n)=a+b+c+\ldots .$$

(Properties 8 and 9 apply.)

(2) Show that from any solution to $x^2+y^2=z^2$ we may find an infinite number. (Property 6 applies.)

(3) Show that $Q(n)$, the number of quadratfrei numbers not exceeding n, satisfies $Q(n)<2^{\pi(n)}$ where $\pi(n)$ is the number of primes not exceeding n. (Property 7 applies.)

(4) Show that $\log p_i/\log p_j$ is irrational. (Consider the constant value contour running through any number, n, in the p_i, p_j plane. It will pass

through no other lattice point since no other point is associated with n. Therefore its gradient is irrational.)

(5) Show that for any two primes p_i and p_j the number of integers of the form $p_i^a \,.\, p_j^b$ not exceeding a given constant C is approximately equal to :

$$\frac{\log\ p_iC.\ \log\ p_jC}{2\log p_i \log p_j}.$$

(Count lattice points bounded by contour through C.)

(6) Consider any n-dimensional rectangular block of points contained by sides $a, b, \ldots, n$ extending positively from a point q. Then, if the sums of the lattice points lying on the sides are $A, B, \ldots, N$ respectively and the number associated with the lattice point q is Q, the sum of all points in the block is given by :

$$\frac{A\,.\,B\,.\ldots.\,N}{Q^{n-1}}.$$

(7) Show, by summing the relevant geometrical series, that the sum of all numbers in the n-octant vertically opposite the principal octant is equal to :

$$\prod_{i=1}^{\infty} (1-p_i^{-1})^{-1}.$$

Alternatively, show that the sum of these numbers is the sum of the reciprocals of the numbers in the principal octant, that is to say : $\sum_{n=1}^{\infty} \frac{1}{n}$.

(8) If $\sigma(n)$ is the sum of the divisors of n, including 1 and n, show by inspection of the lattice that, when n is of the form $p_i^a \,.\, p_j^b$, we have the relationship :

$$\sigma(n) = \sigma(p_i^a) + \sigma(p_j^b) + p_ip_j \,.\, \sigma\left(\frac{n}{p_i \cdot p_j}\right) - 1.$$

(9) Use the result of Problem 6 to obtain the usual expression for $\sigma(n)$ by putting q at the origin.

(10) Show that the number of ways of reducing a number to unity by repeatedly dividing it by its prime factors is equal to the number of routes from n, the point representing the number, to the origin and is, therefore :

$$\frac{(a+b+c+\ldots)!}{a!+b!+c!\ldots},$$

where the number $n = p_i^a p_j^b p_k^c \ldots$.

E. de St. Q. I.

2370. *An invariant of two circles.*

Let S, S' be two circles of radii r, r', and let t, t' be their direct and transverse common tangents. After inversion, let the circles become Σ and Σ', of radii R, R', and let T, T' be their direct and transverse common tangents. When we say that t^2/rr' is invariant, we generally understand that

$$t^2/rr' = T^2/RR', \quad t'^2/rr' = T'^2/RR'.$$

The proof that is usually given is as follows. " If d is the distance between the centres of the circles,

$$(d^2 - r^2 - r'^2)/rr' = -2\cos\theta,$$

where θ is the angle between the circles. But the angle between two curves is unaltered in inversion. Hence $(d^2 - r^2 - r'^2)/rr'$ is invariant. But

$$t^2/rr' = \{d^2 - (r-r')^2\}/rr' = 2 + (d^2 - r^2 - r'^2)/rr',$$

and hence t^2/rr' is invariant."

It is however easy to show analytically that t^2/rr' is not *always* invariant. Taking the centre of inversion as the origin, let the circles be

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0,$$
$$S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0.$$

If p denotes the mutual power of the two circles,

$$p = d^2 - r^2 - r'^2 = c + c' - 2gg' - 2ff';$$

so, as above, $$t^2/rr' = 2 + p/rr'.$$

If k is the constant of inversion, the equations of the inverse circles are

$$\Sigma \equiv x^2 + y^2 + 2(kg/c)x + 2(kf/c)y + k^2/c = 0,$$
$$\Sigma' \equiv x^2 + y^2 + 2(kg'/c')x + 2(kf'/c')y + k^2/c' = 0.$$

If P is the mutual power of these circles, then from the expression for p above, it is easily seen that

$$P = k^2p/cc'.$$

Also,

$$R^2 = k^2(g^2 + f^2 - c)/c^2 = k^2r^2/c^2, \quad \text{and} \quad R'^2 = k^2r'^2/c'^2.$$

Hence

$$RR' = k^2rr'/\sqrt{(c^2c'^2)} = k^2rr'/|\,cc'\,|.$$

Thus

$$P/RR' = (p/rr') \,.\, |\,cc'\,|/cc'.$$

Hence only if c and c' have the same sign do we have

$$P/RR' = p/rr'.$$

In this case,

$$T^2/RR' = 2 + P/RR' = 2 + p/rr' = t^2/rr'$$

and similarly

$$T'^2/RR' = t'^2/rr'.$$

But if c and c' have opposite signs, $P/RR' = -p/rr'$, and we have

$$T^2/RR' = -t'^2/rr', \qquad T'^2/RR' = -t^2/rr'.$$

Hence t^2/rr' is not always an invariant; but $(p/rr')^2$ is an invariant of the two circles.

The mistake is due to the want of precision in the definition of t, the common tangent of the two circles, and in the definition of the angle between two circles. Usually, the angle between two circles is defined as the angle between their tangents. There are two angles, one acute and the other obtuse. Which is the angle to be taken? The conventional answer will be " the acute angle ". But if we define it clearly as the acute angle, this definition will not hold good when the circles do not intersect in real points, as there is no association of magnitude regarding imaginary angles. Thus, the present definition of the angle between two circles (or that between two lines) is left delightfully vague. So I suggest that a more precise method would be to define the angle made by one circle S' with another circle S at a point of intersection P to be the angle made by the tangent PT' to S' with the tangent PT to S, the angle being given by the analytical expression

$$\theta = \tan^{-1}\{(m' - m)/(1 + mm')\},$$

where the principal value of the right-hand side is taken.

The following numerical illustration may be added. Let

$$S \equiv x^2 + y^2 + 4x + 6y - 3 = 0,$$
$$S' \equiv x^2 + y^2 - 4x - 4y + 7 = 0.$$

Here the centres are $(-2, -3)$ and $(2, 2)$, and the radii are 4, 1. Thus

$$t^2/rr' = \{d^2 - (r - r')^2\}/rr' = 8,$$

and

$$t'^2/rr' = \{d^2 - (r + r')^2\}/rr' = 4.$$

Let the circles be inverted with respect to the circle

$$x^2 + y^2 - 1 = 0.$$

The centre of inversion is the origin and the constant of inversion is unity. The centre of inversion is inside S and outside S'. The inverted circles are

$$\Sigma \equiv x^2 + y^2 - \tfrac{4}{3}x - 2y - \tfrac{1}{3} = 0,$$
$$\Sigma' \equiv x^2 + y^2 - \tfrac{4}{7}x - \tfrac{4}{7}y + \tfrac{1}{7} = 0.$$

The centres are $(\tfrac{2}{3}, 1)$ and $(\tfrac{2}{7}, \tfrac{2}{7})$, while the radii are $\tfrac{4}{3}$ and $\tfrac{1}{7}$. Similar calculations to the above give

$$T^2/RR' = -4, \quad T'^2/RR' = -8.$$

Hence

$$T^2/RR' = -t'^2/rr' \quad \text{and} \quad T'^2/RR' = -t^2/rr'.$$

S. SURYANARAYANA IYER.

2371. *An example from electricity.*

Long straight parallel wires, n in number, each carrying the same current C cut a plane perpendicular to them in the vertices $P, Q, \ldots, Z$ of a regular polygon, centre O. Show that the magnetic force at any point A in the plane of the polygon is

$$2nC \,.\, OA^{n-1}/PA \,.\, QA \,.\, \ldots \,.\, ZA.$$

Special forms of this problem, usually for an equilateral triangle or a square, appear in textbooks, but the solution is just as simple in the general case.

Take OP as initial line and let θ be the angle POQ, so that $n\theta = 2\pi$, and writing $OP = a$ we require the modulus of

$$\begin{aligned} w &= 2iC\{(z-a)^{-1} + (z - a \exp i\theta)^{-1} + \ldots + (z - a \exp i(n-1)\theta)^{-1}\} \\ &= 2iC \,.\, nz^{n-1}/(z-a)(z - a \exp i\theta)\ldots(z - a \exp i(n-1)\theta), \end{aligned}$$

the coefficients of $z^{n-2}, z^{n-3}, \ldots, z^0$ are all zero, because

$$a, a \exp i\theta, \ldots, a \exp i(n-1)\theta$$

are the roots of $z^n - a^n = 0$ and hence their sum, the sum of their products two at a time, and so on, all vanish. Thus the magnetic force, which is $|w|$, is

$$2nC \,.\, OA^{n-1}/PA \,.\, QA \,.\, \ldots \,.\, ZA.$$

G. POWER.

2372. *Réciproque du théorème de Pythagore.**

Cette note a pour objet de donner une démonstration de la réciproque du théorème de Pythagore *indépendante* de celui-ci et susceptible de n'utiliser que des théorèmes du Premier Livre pour certains triangles rectangles.

1. *Théorème. Si le carré du nombre qui mesure l'un des côtés est égal à la somme des carrés de ceux qui mesurent les autres, le triangle est rectangle.*

En effet, soit un triangle ABC dont les mesures a, b, c des côtés BC, CA, AB vérifient les équations

$$a^2 = c^2 - b^2 = (c-b)(c+b). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{i})$$

* A ce sujet, voir la note 1036 de N. J. Chignell, *Gazette*, 1932, p. 204.

Sur les côtés BC, AB et sur la droite BA, marquons les points B_1, A_1 et D, E, tels que l'on ait

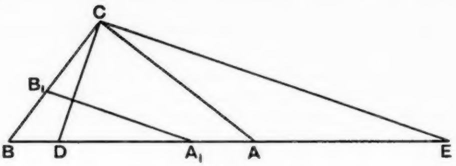


FIG.

$$BB_1 = c - b, \quad BA_1 = BC = a,$$

et

$$AD = -AE = CA = b,$$

ce qui donne

$$BE = BA + AE = c + CA = c + b.$$

Dès lors, d'après (i), on obtient les égalités

$$BB_1/BC = (c-b)/a = a/(c+b) = BA_1/BE, \quad \text{......................(ii)}$$

qui établissent le parallélisme des droites A_1B_1 et EC et, par suite l'égalité des angles (A_1B_1, A_1B) et (EC, EA). D'autre part, les triangles BA_1B_1 et CBD ayant un angle égal en B compris entre côtés égaux

$$BC = BA_1, \quad BB_1 = c - b = BA - DA = BD,$$

sont égaux. Le triangle CAE est isoscèle par construction. En définitive, des égalités d'angles

$$(CB, CD) = (A_1B_1, A_1B) = (EC, EA) = (CA, CE) = \theta, \quad \text{...............(iii)}$$

qui proviennent de ces remarques, il résulte que les angles (CB, CA) et (CD, CE) peuvent coïncider par une rotation de l'angle θ autour du point C et le premier d'entre eux est égal au second *qui est droit*.* C. Q. F. D.

2. Cette démonstration utilise la similitude (réciproque du théorème attribué à Thalès†). Mais, si A_1 et B_1 sont des points homologues pris parmi ceux qui divisent les segments rectilignes BE et BC en m parties *égales*, m étant un nombre entier arbitraire, le parallélisme des droites A_1B_1 et EC, et par suite les égalités d'angles (iii), peuvent s'établir en usant seulement de théorèmes du Premier Livre. C'est le cas des triangles dont les mesures a, b, c des côtés sont des nombres premiers entre eux deux à deux vérifiant, à la fois, les relations

$$a^2 = c^2 - b^2 = (c-b)(c+b), \quad c - b = 2. \quad \text{................(iv), (v)}$$

Car $c + b = \frac{1}{2}a^2$, et les segments $BB_1 = 2$ et $DA_1 = a$ sont égaux à la $(\frac{1}{2}a)^e$ partie de BC et de BE, $\frac{1}{2}a$ étant un nombre entier, d'après (iv), (v).

Exemple. $a = 12$, $b = 35$, $c = 37$; $\frac{1}{2}a = 6 = a/(c+b)$.

Plus particulièrement, si

$$a = 2^n \quad \text{et} \quad c - b = 2, \quad (n > 1), \quad b = 2^{2(n-1)} - 1, \quad c = 2^{2(n-1)} + 1,$$

les segments BB_1 et BA_1 équivalent à la $(2^{n-1})^e$ partie de BC et de BE.

Exemples. $n = 2$; $a = 4$, $b = 3$, $c = 5$:
$n = 3$; $a = 8$, $b = 15$, $c = 17$.

* On démontre par des théoremès du Premier Livre que le triangle CDE dont la médiane CA est égale à la moitié du côté DE est rectangle.

† Thalès de Milet, 639–548 avant l'ère chrétiénne.

Enfin, c'est aussi le cas des triangles où l'on a, à la fois,

$$a^2 = c^2 - b^2 = (c-b)(c+b), \qquad c - b = 1.$$

Car $c + b = a^2$, et les segments BB_1 et BA_1 sont égaux a là a^e partie de BC et de BE.

Exemple. $a = 19$, $b = 180$, $c = 181$; $c + b = 180 + 181 = 19^2$.

Dans tous ces cas particuliers, il est évident que les nombres a, b, c premiers entre eux deux à deux, peuvent être remplacés par des multiples quelconques de ces nombres.
VICTOR THÉBAULT.

2373. *A rough census of prime numbers.*

The prime number theorem shows that the number of primes less than n approximates, when n is very large, to the ratio of n to its natural logarithm. It is clear from the tables that if we denote this number of primes by $P(n)$, then for values of n greater than 10,

$$P(n) > n/\log n.$$

Hardy and Wright, *Introduction to the theory of numbers*, p. 9, state that the ratio of $P(n)$ to $n/\log n$ takes the values 1·159... , 1·084... , 1·053... when n is successively 10^3, 10^6, 10^9. These figures suggest that the error is larger than 5% until n exceeds 10^9. For much smaller values of n the integer I nearest to the fraction $n/2 \log_{10} n$ provides a good approximation. We have in fact the following table of values showing the percentage error E to the nearest integer.

n	10	10^2	2.10^2	3.10^2	4.10^2	5.10^2	10^3
$P(n)$	4	25	46	62	78	95	168
I	5	25	43	61	77	93	167
E	+25	0	−7	−1	−1	−2	−1

n	2.10^3	3.10^3	4.10^3	5.10^3	10^4	2.10^4	10^5
$P(n)$	303	430	550	669	1229	2260	9592
I	303	431	555	675	1250	2325	10000
E	0	+0	+1	+1	+2	+3	+4

When n is 10^6, 10^7, 10^9, E has the values $+6$, $+7$, $+9$. These percentages suggest that in the range

$$2.10^4 \leqslant n \leqslant 10^9$$

a better approximation would be $I - \frac{1}{200}n$,

or better still,

$$I - \frac{n}{200 + \log_{10} n}.$$

B. E. LAWRENCE.

2374. *On the Equation $x^2 - cy^2 = z^2$.*

The general method of solving $x^2 - cy^2 = \pm h$ is fully discussed in some of the older text-books,* but this type of equation is of comparatively little interest nowadays and the method of solution is probably only known to those who are interested in elementary Diophantine problems. It still, however, provides a useful source of numerical problems for mathematical crossword puzzles and the like.

Solutions of $\qquad x^2 - cy^2 = \pm h$ ....(1)

* *E.g.* Chrystal, *Algebra*, II, XXXIII.

with integral x, y, where c is not a perfect square, depend on the Pellian equations

$$x^2 - cy^2 = \pm 1. \quad \text{......................................(2)}$$

The problem of finding solutions* of (2) was originally proposed by Fermat as a challenge to English mathematicians, and solutions were found by Brouncker and Wallis. The more general problem of solving (1) was first completed by Lagrange. Solutions of (1) when $h < \sqrt{c}$ can be expressed in terms of those of (2); and when $h > \sqrt{c}$ the process is more laborious, depending on reducing the equation to dependence on $X^2 - cY^2 = \pm h_r$ where $h_r < \sqrt{c}$. This method of solution, which finds *all* the integral solutions of (1), is fully described by Chrystal.

In the case when $h = z^2$, there are simpler methods of solution. These are particularly useful when one is interested in finding *some*, rather than *all*, solutions.

No originality is claimed for any of the methods. A glance at the long chapter on the Pell Equation in Dickson's *History of the Theory of Numbers* shows the vast number of papers published on it since 1770† and in view of the formidable array of references, it is scarcely likely that there is anything new to be said; but the methods of solution described here may be of some interest to readers.

The solutions of (2) are provided by the successive convergents in the continued fraction for $\sqrt{c}$. It can be shown that the equation

$$x^2 - cy^2 = 1 \quad \text{......................................(3)}$$

always admits of an infinite number of solutions and

$$x^2 - cy^2 = -1 \quad \text{......................................(4)}$$

does also if the number of quotients in the period of $\sqrt{c}$ is odd, otherwise it has no solutions. In particular, if $c = 2$,

$$\sqrt{2} = 1 + \frac{1}{2+}\,\frac{1}{2+}\,\ldots$$

and the successive convergents p_n/q_n are

$$\tfrac{1}{1},\ \tfrac{3}{2},\ \tfrac{7}{5},\ \tfrac{17}{12},\ \tfrac{41}{29},\ \ldots$$

a table readily extended, since $p_n = 2p_{n-1} + p_{n-2}$ (and q_n similarly). The even convergents provide solutions (x/y) of (3) and the odd convergents solutions of (4) when $c = 2$. Similar methods apply for other values of c.

The equation $x^2 - cy^2 = z^2$.

It is readily seen that solutions of

$$x^2 - cy^2 = z^2 \quad \text{......................................(5)}$$

are given by $z = cm^2 - n^2$, $y = 2mn$, $x = cm^2 + n^2$, for integral m, n. Also, if x, y, z is any one solution, for given z, then further solutions are given by

$$x_1 = ax \pm cby, \qquad y_1 = bx \mp ay,$$

where a, b are solutions of $x^2 - cy^2 = 1$. For,

$$(ax \pm cby)^2 - c(bx \mp ay)^2 = z^2,$$

if

$$(a^2 - cb^2)(x^2 - cy^2) = z^2,$$

which reduces to (5) if $a^2 - cb^2 = 1$.

* "Solutions" will mean "integral solutions" in what follows.

† The first known problem which was, in effect, a solution of the Pell equation dates from 400 B.C.

If $c=2$, $a=3$, $b=2$ and so further solutions of $x^2-2y^2=z^2$, for given z, are $3x \pm 4y$, $2x \pm 3y$.

If $c=3$, $a=2$, $b=1$, giving $2x \pm 3y$, $x \pm 2y$.

If $c=5$, $a=9$, $b=4$, giving $9x \pm 20y$, $4x \pm 9y$.

The solutions of

$$x^2 - cy^2 = -z^2 \qquad \text{.................................(6)}$$

are similarly $x = ax \pm cby$, $y = bx \mp ay$, where (a, b) is a solution of $x^2 - cy^2 = -1$.

In the special case when $c=2$, since $(1, 1)$ is a solution of $x^2 - cy^2 = -1$, the solutions of

$$x^2 - 2y^2 = -z^2 \qquad \text{.................................(7)}$$

can be expressed very simply in terms of those of

$$x^2 - 2y^2 = z^2 \qquad \text{.................................(8)}$$

for, if (p, q) is any solution of (8), $(p \pm 2q, p \pm q)$ is a solution of (7)

Alternatively, the solutions of (7) can be expressed as

$$x = 4mn - (2m^2 + n^2), \quad y = 2m^2 + n^2 - 2mn, \quad z = 2m^2 - n^2,$$

for any integral m, n.

Another method* of deducing the solutions of

$$x^2 - cy^2 = \pm h \text{.................................(9)}$$

from those of (3) consists of modifying the convergents in the continued fraction for $\sqrt{2}$.

Write

$$x_1 = 1 + \frac{a}{b}, \quad x_2 = 1 + \frac{1}{2+(a/b)}, \quad x_3 = 1 + \frac{1}{2+}\;\frac{1}{2+(a/b)},$$

and so on, whence

$$\frac{p_1}{q_1} = \frac{a+b}{b}, \quad \frac{p_2}{q_2} = \frac{3b+a}{2b+a}, \ldots$$

and successive convergents can be calculated from $p_n = 2p_{n-1} + p_{n-2}$ (similarly for q_n).

We get

$$\frac{a+b}{b}, \quad \frac{a+3b}{a+2b}, \quad \frac{3a+7b}{2a+5b}, \quad \frac{7a+17b}{5a+12b}, \quad \frac{17a+41b}{12a+29b}, \text{ ....(10)}$$

Now, a and b can be chosen to have any integral values and solutions of (9), for different values of h, can easily be determined. $a=0$, of course, corresponds to $h=+1$, the odd terms in the sequence (10) corresponding to $h=-1$, the even terms to $h=+1$.

E.g. put $a=2$, $b=1$ and the sequence (10) becomes

$$\tfrac{3}{1}, \ \tfrac{5}{4}, \ \tfrac{13}{9}, \ \tfrac{31}{22}, \ \tfrac{75}{53}, \ldots \text{ ....(11)}$$

Since $(3, 1)$ is easily seen to satisfy

$$x^2 - 2y^2 = 7 \qquad \text{.................................(12)}$$

the odd terms of (11) give solutions of (12) and the even terms give solutions of $x^2 - 2y^2 = -7$.

$a=3$, $b=4$ gives similarly the solutions of

$$x^2 - 2y^2 = \pm 17,$$

for $\dfrac{a+b}{b} = \dfrac{7}{4}$ and $(7, 4)$ satisfies $x^2 - 2y^2 = 17$.

* Communicated to me by Mr. J. Thomas, 9 College Rd., Bangor.

Solutions of

$$x^2 - cy^2 = uv \qquad \text{.......(13)}$$

can be deduced from those of

$$x^2 - cy^2 = u, \qquad \text{.......(14)}$$

and

$$x^2 - cy^2 = v, \qquad \text{.......(15)}$$

for, if (x_u, y_u), (x_v, y_v) are solutions of (14), (15) ; since

$$(x_u x_v \pm c y_u y_v)^2 - c\,(x_u y_v \mp y_u x_v)^2 = (x_u^2 - c y_u^2)(x_v^2 - c y_v^2) = uv,$$

solutions of (13) are

$$\left.\begin{aligned} x &= x_u x_v \pm c y_u y_v \\ y &= x_u y_v \pm y_u x_v \end{aligned}\right\}. \qquad \text{.......(16)}$$

If $c = 2$, $u = 4$, $v = 7$, we see that $x_u = 6$, $y_u = 4$; $x_v = 3$, $y_v = 1$ and (16) gives

$$x = 18 \pm 8 = 26 \quad \text{or} \quad 10,$$
$$y = 6 \pm 12 = 18 \quad \text{or} \quad -6,$$

and (26, 18) and (10, 6) are readily verified as solutions of

$$x^2 - 2y^2 = 28.$$

E. G. PHILLIPS.

2375. *Instantaneous and zero-acceleration centres.*

Elementary vector methods can be used in the following way to obtain the fundamental properties of the instantaneous centre and the zero-acceleration centre for the motion of a plane lamina in its own plane.

1. If $\mathbf{a}$ be the position vector of a point A fixed in the lamina relative to a point O fixed in space, the velocity of any point P of the lamina is

$$\dot{\mathbf{a}} + \boldsymbol{\omega} \times (\mathbf{r} - \mathbf{a}),$$

where $\mathbf{r}$ is the position vector of P with respect to O and $\boldsymbol{\omega}$ is the angular velocity of the lamina at the same instant. The position vector $\mathbf{r}$ of the instantaneous centre, I, will be given by

$$\dot{\mathbf{a}} + \boldsymbol{\omega} \times (\mathbf{r} - \mathbf{a}) = 0, \qquad \text{.......(i)}$$

and on vector multiplication by $\boldsymbol{\omega}$ we have

$$\mathbf{r} = \mathbf{a} + (\boldsymbol{\omega} \times \dot{\mathbf{a}})/\omega^2,$$

giving the locus of I in space (the space-centrode).

The position vector $\mathbf{R}$ of I relative to A is, by (i),

$$\mathbf{R} = (\boldsymbol{\omega} \times \dot{\mathbf{a}})/\omega^2, \qquad \text{.......(ii)}$$

which is the locus of I in the body (the body-centrode), but referred to axes the directions of which are fixed in space. These axes are moving in the body, but we can refer the motion of I in the body to them by using the formula for the rate of change of a vector with respect to moving axes. Remembering that at the instant under consideration these axes are rotating with angular velocity $-\boldsymbol{\omega}$ relative to axes fixed in the body, the velocity of I in the body is

$$\begin{aligned} \dot{\mathbf{R}} + (-\boldsymbol{\omega} \times \mathbf{R}) &= \frac{d}{dt}\left(\frac{\boldsymbol{\omega} \times \dot{\mathbf{a}}}{\omega^2}\right) - \boldsymbol{\omega} \times \left(\frac{\boldsymbol{\omega} \times \dot{\mathbf{a}}}{\omega^2}\right) \quad \text{from (ii)} \\ &= \frac{d}{dt}\left(\frac{\boldsymbol{\omega} \times \dot{\mathbf{a}}}{\omega^2}\right) + \dot{\mathbf{a}}, \\ &= d\mathbf{r}/dt, \end{aligned}$$

which, from (i), is the velocity of I in space and the well-known property of these centrodes follows at once.

2. Let $\boldsymbol{\rho}$ be the position vector of any point P of the lamina relative to A. The velocity of P relative to A will be $\boldsymbol{\omega} \times \boldsymbol{\rho}$ and using again the result for the rate of change of a vector referred to moving axes, remembering this time that the axes fixed in the body are rotating with angular velocity $\boldsymbol{\omega}$, the acceleration of P relative to A will be

$$\frac{d}{dt}(\boldsymbol{\omega} \times \boldsymbol{\rho}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}),$$

which can be written

$$\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} - \omega^2 \boldsymbol{\rho}$$

since ρ is fixed in the body. The acceleration of P in space will therefore be

$$\ddot{\mathbf{a}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} - \omega^2 \boldsymbol{\rho},$$

so that the position vector $\boldsymbol{\rho}$ of the centre of zero acceleration, J, will be given by

$$\ddot{\mathbf{a}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} - \omega^2 \boldsymbol{\rho} = 0$$

or

$$\omega^2 \boldsymbol{\rho} - \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} = \ddot{\boldsymbol{\alpha}}. \qquad \text{(iii)}$$

Forming (a) the scalar product of each side of (iii) with itself, we have

$$(\omega^4 + \dot{\omega}^2)\rho^2 = \ddot{a}^2$$

and (b) the scalar product of (iii) with $\boldsymbol{\rho}$, we have

$$\ddot{\mathbf{a}}\boldsymbol{\rho} = \omega^2 \rho^2.$$

These two equations now fix the position of J at any instant.

A. BUCKLEY.

2376. *Pascal's theorem : for the collector of projective proofs.*

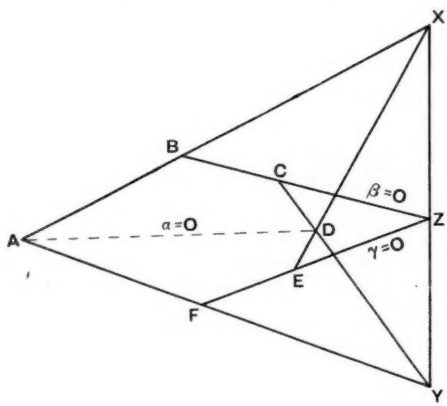


FIG.

A, B, C, D, E, F are six points on a conic $s = 0$. The line-pairs AB, CD and AF, ED have equations $s = \alpha\beta$, $s = \alpha\gamma$. The points X and Y lie on both line-pairs, and not on $\alpha = 0$, and so they lie on $\beta = \gamma$, which passes through Z.

A. R.

2377. *The computation of Fourier coefficients.*

The coefficients a_k and b_k $(k \geqslant 1)$ of a Fourier series

$$\sum_k \{a_k \cos kx + b_k \sin kx\}$$

representing a piece-wise regular function $y(x)$, with a period 2π, can be determined by simple differentiation by means of the relations

$$\pi k a_k = [Cs + Sc]_1^2, \quad \pi k b_k = [-Cc + Ss]_1^2,$$

where

$$C = \sum_0^\infty (-1)^\nu y^{(2\nu)}/k^{2\nu}, \quad S = \sum_0^\infty (-1)^\nu y^{(2\nu+1)}/k^{2\nu+1},$$

$$s = \sin kx, \quad c = \cos kx, \quad y^{(\nu)} = d^\nu y/dx^\nu,$$

all the expressions in the square brackets being taken between the limits x_1 and x_2. These relations are obtained by repeated partial integration of the usual equations for a_k and b_k. Thus, with the exception of the constant $\frac{1}{2}a_0$, all the components of a Fourier series may be determined without integration of the primitive function $y(x)$.

The value of $r_k = \sqrt{(a_k^2 + b_k^2)}$ is given by

$$\pi k r_k = \sqrt{\{R_1^2 + R_2^2 - 2(r_1 r_2)(R_1 R_2) + 2[r_1 r_2][R_1 R_2]\}}$$

where

$$R_1^2 = C_1^2 + S_1^2, \quad R_2^2 = C_2^2 + S_2^2,$$

$$(r_1 r_2) = (c_1 c_2 + s_1 s_2), \quad (R_1 R_2) = (C_1 C_2 + S_1 S_2),$$

$$[r_1 r_2] = (s_1 c_2 - s_2 c_1), \quad [R_1 R_2] = (S_1 C_2 - S_2 C_1).$$

The phase angle is $\phi_k = \tan^{-1}(b_k/a_k)$ or with $t = s/c$

$$\tan \phi_k = \frac{(S_2 t_2 - C_2) - (S_1 t_1 - C_1) c_1/c_2}{(S_2 + C_2 t_2) - (S_1 + C_1 t_1) c_1/c_2}.$$

The relations for the coefficients $a_k^{(-n)}$ and $b_k^{(-n)}$ of the primitive function integrated n times, between the same limits x_1 and x_2, are easily obtained and require n integrations only.

The procedure of giving the values of a_k and b_k with the limits already inserted in the corresponding expressions has the disadvantage of not allowing the values of the coefficients for other limits to be found without going through the entire calculation again. It is therefore preferable in many cases that the expressions for a_k and b_k be given with the limits left indeterminate. For example, instead of writing $a_k = 4(-1)^k/k^2$ it is more convenient for further computations to write

$$a_k = \frac{1}{\pi k}\left[\left(x^2 - \frac{2}{k^2}\right)\sin kx + \frac{2x}{k}\cos kx\right]_1^2$$

and putting $x_1 = -\pi$ and $x_2 = +\pi$ gives the previous expression.

E. J. Nestorides.

2378. *An approximate construction for π.*

The following method (in which the accuracy of the result is limited only by one's ability to draw accurately) makes use of the formula

$$\pi = 2 \sec \frac{\pi}{2^2} \sec \frac{\pi}{2^3} \sec \frac{\pi}{2^4} \dots \text{ ad inf.}$$

(obtained by putting $x = \frac{1}{2}\pi$ in example 7, page 114, of Bromwich's *Theory of Infinite Series*, 1926).

Draw a line OA with length 2 units, and draw OB at right angles to it. Bisect $\angle AOB$ by OA_1, bisect $\angle A_1OB$ by OA_2, bisect $\angle A_2OB$ by OA_3, etc.

Draw AP_1 perpendicular to OA meeting OA_1 at P_1,
P_1P_2 perpendicular to OA_1 meeting OA_2 at P_2,
P_2P_3 perpendicular to OA_2 meeting OA_3 at P_3, etc.

Then the lengths of OP_1, OP_2, OP_3, etc., which are

$$2\sec\frac{\pi}{2^2},\quad 2\sec\frac{\pi}{2^2}\sec\frac{\pi}{2^3},\quad 2\sec\frac{\pi}{2^2}\sec\frac{\pi}{2^3}\sec\frac{\pi}{2^4},\ \text{etc.},$$

approximate rapidly to π units. The reader may supply his own diagram.

Which approximation one uses depends on the degree of accuracy required; the calculated errors are:

OP_1,	error 10%
OP_2	2·5%
OP_3	0·6%
OP_4	0·2%
OP_5	0·04%
OP_6	0·01%
OP_7	0·003%
etc.	

J. G. FREEMAN.

2379. *Construction of cubic expressions.*

How can we form a cubic that not only vanishes but is also stationary for rational values of x?

(A) It will be sufficient to consider a cubic

$$x(x-a)(x-b),$$

(where a and b are rational numbers) which is seen to be stationary when

$$3x^2-2(a+b)x+ab=0.$$

If this quadratic is to have rational roots, its discriminant must be a perfect square, namely,

$$(a+b)^2-3ab=a^2-ab+b^2=c^2,$$

where c is a rational number.

Thus $$b(b-a)=(c+a)(c-a)$$

and so $$\frac{b-a}{c+a}=\frac{c-a}{b}=\frac{p}{q},$$

where p and q are rational numbers.

Hence $$(p+q)a-qb+pc=0,$$

and $$qa+pb-qc=0,$$

giving $$\frac{a}{q^2-p^2}=\frac{b}{q^2+2pq}=\frac{c}{p^2+pq+q^2}.$$

The required cubic is then

$$x(x-q^2+p^2)(x-q^2-2pq).$$

An equally good solution is

$$(x-p^2)(x-q^2)[x-(p+q)^2].$$

(B) The easily verified fact that one of the stationary values of the last cubic is $-p^2q^2(p+q)^2$ suggests another method of approach. If the equation

$$(x-p^2)(x-q^2)(x-r^2)=-p^2q^2r^2$$

has a pair of equal roots $x=x_1$, other than the zero root, it is clear that $x=x_1$ gives a stationary value of the left-hand side. Now,

$$2x_1=\text{sum of the roots}=p^2+q^2+r^2,$$

which is rational if p, q and r are.

Our condition is then that

$$x^2-(p^2+q^2+r^2)x+q^2r^2+r^2p^2+p^2q^2$$

should be a perfect square.

Thus $$(p^2+q^2+r^2)^2-4(q^2r^2+r^2p^2+p^2q^2)=0,$$

which, after some reduction, becomes

$$[r^2-(p+q)^2][r^2-(p-q)^2]=0,$$

and the required condition is that $r=p+q$, since we can assume that p, q, r are all positive and in ascending order of magnitude.

(C) The most direct way of finding the required cubic is to utilise the fact

that $$x^3-3hx^2+k$$

is clearly stationary for $x=0$ and $x=2h$.

Thus, all we have to do is to write down a cubic with rational roots which lacks a term in x.

This is $$(x-\alpha)(x-\beta)\left(x+\frac{\alpha\beta}{\alpha+\beta}\right),$$

which can easily be verified to be equivalent to the forms given above.

C. M. SEGEDIN.

2380. *Conormal points on an ellipse.*

It is well known that in general four normals can be drawn through a given point to an ellipse. In this note we consider the reality of these normals and deduce by a method that does not involve too much tiresome algebra, the result that if the given point lies inside the evolute of the ellipse there are four real normals, and that there are only two real normals if the point lies outside. This result is not new but the method of obtaining it may be.

The feet of the normals from the point $(\bar{x}, \bar{y})$ are given by the intersection of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad \ldots\ldots\ldots\ldots(1)$$

and the Apollonian Hyperbola

$$\frac{x-\bar{x}}{x/a^2}=\frac{y-\bar{y}}{y/b^2}. \quad \ldots\ldots\ldots\ldots(2)$$

If each of these two fractions in (2) is set equal to λ, we find that

$$x=\frac{a^2\bar{x}}{\lambda-a^2} \quad \text{and} \quad y=\frac{b^2\bar{y}}{\lambda-b^2}, \quad \ldots\ldots\ldots\ldots(3)$$

and hence from (1)

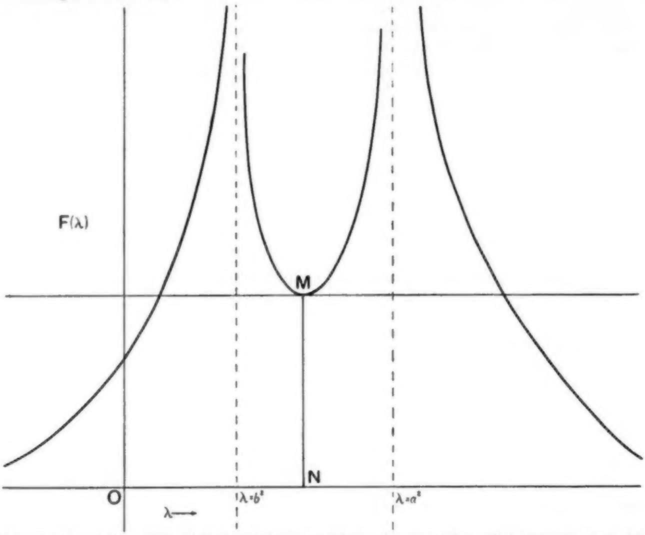
$$\frac{a^2\bar{x}^2}{(\lambda-a^2)^2}+\frac{b^2\bar{y}^2}{(\lambda-b^2)^2}=1. \quad \ldots\ldots\ldots\ldots(4)$$

The roots of this quartic in λ give by means of (3) the feet of the normals sought.

Consider the graph of

$$F(\lambda)\equiv\frac{a^2\bar{x}^2}{(\lambda-a^2)^2}+\frac{b^2\bar{y}^2}{(\lambda-b^2)^2}. \quad \ldots\ldots\ldots\ldots(5)$$

Clearly $F(\lambda)$ is always positive, and for large λ it behaves like $1/\lambda^2$. There are vertical asymptotes at $\lambda = a^2$ and $\lambda = b^2$. The graph is thus as shown, with a minimum point at M.



The roots of (4) are obtained by finding the points of intersection of this graph with a line parallel to the λ-axis and distant unity above it. If MN is the minimum value of $F(\lambda)$, it is clear that there are four points of intersection if $MN < 1$ and only two if $MN > 1$.

Now $F'(\lambda)$ vanishes when

$$\frac{a^2\bar{x}^2}{(\lambda - a^2)^3} + \frac{b^2\bar{y}^2}{(\lambda - b^2)^3} = 0\,;$$

that is, when

$$\frac{(a\bar{x})^{2/3}}{\lambda - a^2} = -\frac{(b\bar{y})^{2/3}}{\lambda - b^2} = -\frac{(a\bar{x})^{2/3} + (b\bar{y})^{2/3}}{a^2 - b^2}\,.$$

Substitution of these values of $1/(\lambda - a^2)$ and $1/(\lambda - b^2)$ in (5) will give the stationary value,

$$MN = \frac{[(a\bar{x})^{2/3} + (b\bar{y})^{2/3}]^3}{(a^2 - b^2)^2}\,.$$

Now, we have two equal roots when $MN = 1$, that is, when

$$(a\bar{x})^{2/3} + (b\bar{y})^{2/3} = (a^2 - b^2)^{2/3}.$$

This is the condition that $(\bar{x}, \bar{y})$ should lie on the evolute of the ellipse (as we should expect).

We have four real roots when $MN < 1$, that is, when

$$(a\bar{x})^{2/3} + (b\bar{y})^{2/3} < (a^2 - b^2)^{2/3},$$

so that $(\bar{x}, \bar{y})$ must lie inside the evolute in this case.

And we have only two real roots when $(\bar{x}, \bar{y})$ lies outside the evolute.

C. M. SEGEDIN.

2381. *The binomial theorem for negative integers.*

Pascal's Triangle has long been used to give the coefficients in the expansion of the Binomial theorem for positive integral indices. But if we construct a Pascal triangle :

1					
1	1				
1	2	1			
1	3	3	1		
1	4	6	4	1	
1	5	10	10	5	1
1	6	15	20	...	

it will be seen that the vertical columns give the coefficients for the expansion of the Binomial Theorem for negative integral indices. I have not seen any reference to this simple and useful fact.*

B. COHEN.

2382. *Grid references.*

1. At first sight the O.S. Grid System appears to be a normal system of rectangular coordinates. Closer examination discloses two differences in the nature of the approximation to definition of position ; one of method, the other of magnitude.

2. Regarded as an approximation, coordinates (41, 36) indicate that a point lies within a square of side 1 unit having (41, 36) as its central point, the exact coordinates each having a possible difference of $\pm \cdot 5$ units.

3. On the other hand, a grid reference 4136 indicates that a point lies within a square of side 1 unit having 4136 as its SW corner, each half of the exact reference having a possible difference of $+1$ unit.

4. It appears then that not only is the approach to the limit different but the margin of error in a grid reference is twice that in the normal coordinate system.

I. FITZROY JONES.

2383. *A formula for rational right-angled triangles.*

The problem of finding rational Pythagorean triangles is a historic one, for D. E. Smith (I 281, II 290) gives formulae for these triangles from Plato, Proclus, and probably Pythagoras (*c* 540 B.C.), and from Bhaskara later. The subject was dealt with practically in *M.G.*, July 1938 and July 1939. The present writer has found it useful to have a number of such triangles for use with pupils testing Pythagoras' Theorem, or drawing rectangles and testing the accuracy of the drawing by measuring the diagonals. There is a rule for finding these triangles, apparently unknown, which is much simpler than those referred to above, *viz.* :

" Take any two rational numbers whose product is 2, and add 2 to each. The results are the perpendicular sides of a rational right-angled triangle."

Ex. $\frac{13}{17} \times \frac{34}{13} = 2$. The sides are $\frac{47}{17}$, $\frac{60}{13}$; or, clearing fractions, 611 and 1020. The hypotenuse is 1189.

Proof of Rule.

For *every* right-angled triangle with *rational* sides there is some rational number n such that

$$\begin{matrix}\text{(Number of units in the} \\ \text{perimeter of the triangle)}\end{matrix} = \begin{matrix}\text{(number of square units in the} \\ \text{area)} \times 2n.\end{matrix}$$

* A somewhat similar arrangement is given in Note 1519 (XXV, p. 118).

Taking x, y, as perpendicular sides,

$$x + y + \sqrt{(x^2 + y^2)} = nxy.$$

Thus
$$(x + y - nxy)^2 = x^2 + y^2,$$
$$n^2xy - 2n(x + y) + 2 = 0,$$

Therefore
$$\left.\begin{aligned} (nx-2)(ny-2) &= 2 = z \times \frac{2}{z} \\ nx = 2 + z, \quad ny &= 2 + \frac{2}{z} \\ x : y = (2+z) &: \left(2 + \frac{2}{z}\right) \end{aligned}\right\}, \text{ where } z \text{ is } \textit{any} \text{ rational number.}$$

This proof is a simple piece of work which would interest many pupils in the higher forms of the secondary school, and might be discovered by some pupils. But it can be further simplified for more pupils by making n equal to $\frac{1}{2}$. The problem then becomes,

"To find a triangle which contains as many square units as there are units in the perimeter."

The ultimate equation is then;

$$(x - 4)(y - 4) = 8.$$

Taking only whole number factors of 8, the roots are:

$$x = 4 + 1, \quad y = 4 + 8, \quad \textit{i.e. } \text{triangle } 5/12/13,$$
$$x = 4 + 2, \quad y = 4 + 4, \quad \textit{i.e. } \text{triangle } 6/8/10.$$

These are the only two whole number solutions. R. S. Williamson.

CORRESPONDENCE.

THE NAME OF THE GAME OF NIM.

To the Editor of the *Mathematical Gazette*.

Sir,—On p. 119 of the May, 1953, issue of the *Mathematical Gazette* I read the conjecture by Professor Alan S. C. Ross that Professor C. L. Bouton derived the name of the game "Nim" from the imperative singular of the German verb *nehmen*. I am writing to verify this conjecture. Professor Bouton was a teacher of mine, and later I became his colleague. I was a frequent visitor in his home. He had had numerous mathematical and personal contacts with Germany, and received his Ph.D. from Leipzig in 1898. I distinctly recall his saying to me that he had chosen the name from the German word "nimm", as a word that might well be used frequently during the play of the game, but had dropped the final "m".

Yours, etc.,

University of Harvard. J. L. Walsh

1754. Launching by catapult and the use of arrester gear followed standard aircraft-carrier practice, but the idea of placing on the ground, instead of in the aircraft, the means of absorbing the vertical kinetic energy on landing was entirely new.—*The Times*, report of Sir Ben Lockspeiser's British Association address, September 1952. [Per Mr. F. Puryer White.]

REVIEWS

The Collected works of C. J. Keyser. II. The rational and the superrational. Pp. viii, 259. $4.25. 1952. (*Scripta Mathematica*, 186th Street, Amsterdam Avenue, New York)

In this volume, the editor, Professor Ginsburg, has collected the essays in which Keyser brought mathematical facts, methods and ways of thinking to bear on problems of philosophy and religion. While we may at times doubt the relevance of the analogies or the legitimacy of applying mathematical processes to points of morality, we must admire the evident sincerity of purpose and the resolute effort to penetrate deep human problems by the clarity of mathematical thinking. The whole tone of the volume can be described by quotation from the essay on " The role of mathematics in the tragedy of our modern culture ". This tragedy, says Keyser, is the fact " that among the major components of our modern culture there are two outstanding and very precious components which have been and are so inaccessible, so remote from the ordinary thinking of man, as to be virtually unknown save to a small number of specialists ". These components are the literature of mathematics and the literature of mathematical physics. " Though the tragedy cannot be ended, it can be greatly mitigated . . . by enabling the great class of intellectual non-mathematicians to become intelligent about mathematics without having to become mathematicians ". Much is being done to make this possible ; Keyser mentions a dozen books which provide clear and simple expositions of great mathematical doctrines, and many more could be added to his list. The need is great ; if we comprehend the bases of modern civilisation, we may even be able to save it.

T. A. A. B.

Smithsonian Logarithmic Tables to Base e and Base 10. By G. W. Spenceley, Rheba M. Spenceley, and E. R. Epperson. Pp. xiii, 402. $4.50. 1952. Smithsonian Miscellaneous Collections, Vol. 118. (Smithsonian Institution, Washington)

This most useful volume was planned by Professor and Mrs. Spenceley, of Miami University, Oxford, Ohio, who are already well known as the authors of the fine *Smithsonian Elliptic Functions Tables* (Smithson. Misc. Coll., Vol. 109, 1947). The work of preparation was greatly helped by four named Miami students, including Mr. Epperson, who contributes the Introduction. Natural logarithms (pages 1–200) and common logarithms (pages 203–402) are tabulated in almost identical form. In both tables the logarithms are to 23 decimals, and the numbers whose logarithms are given, in three parallel columns, are N, $1+N\,.\,10^{-7}$, and $1+N\,.\,10^{-11}$, where N takes all integral values 1 to 10,000. The only difference in form is the very natural one that $\ln N$ is given in full while the characteristic of $\log N$ is omitted. Examples of the use of the table in finding logarithms and antilogarithms, both natural and common, are worked out in the Introduction.

The values of $\ln N$ were taken or computed from the 48-decimal table of the great calculator Wolfram, in the version by Vega in his *Thesaurus* of 1794. This gives natural logarithms of all integers up to 2201 and then of all primes (and more than 500 composites) up to 10,009. The few known errors which affect decimals relevant to the computation were corrected. The other two columns of natural logarithms were computed from the series for $\ln(1+x)$. The values of the common logarithms were deduced by multiplication by the modulus. All computations were performed to 28 decimals on hand-operated machines and checked by differencing.

The printing is entirely adequate for practical use, though the spacing of the

equal-height digits is sometimes slightly irregular and adjacent digits are very occasionally just in contact.

There are extremely few tables of either natural or common logarithms which are comparable in importance with those under review, and I can think of none which unite such powerful tools of each kind in a single handy volume. The modest hope that the tables will be " a welcome addition to the existing tables of logarithms " will surely be amply fulfilled. Only with the advent of calculating machines has mathematics found effective successors, among whom the Spenceley group must now be included, to heroic logarithmic calculators such as Abraham Sharp and Isaac Wolfram. On most practical occasions, computers will gladly forgo half of Wolfram's 48 decimals in exchange for the convenience of having all integral arguments up to 10,000 and also the associated radix tables. At a late date in the history of logarithmic tables, the authors have appreciated the existence of a gap and have filled it in praiseworthy fashion. All computing establishments will naturally obtain the volume, while the moderate price keeps it within the reach of individual workers.

A. FLETCHER.

Geschichte der Mathematik. By O. BECKER & J. E. HOFMANN. DM. 10. 1951. (Athenäum Verlag, Bonn)

This important little book, crowded with interesting and very pertinent detail, is characterised first and foremost by an almost pedantic regard for historic accuracy. This means that, in the interest of brevity, it is strongly focussed on the written records, with the express purpose of aiding the partly informed reader to acquaint himself with the particular sources that most concern him. In this respect, the work may be described as a vivid textual history of mathematics. The four-paged preface, itself a masterpiece of succinctness, gives its due to what is called " latent mathematics ", discernible not only in the practical techniques of mensuration, architecture and commerce and in the astronomical divisions of time, but in speech formations and constructions, logical concatenations, state and social systems, usages and cults, art and ornamentation, dance and music. " Scientific mathematics ", on the other hand, separates out as soon as explicit mathematical statements are available, not before ; and it is substantial to the same extent as they are (the only true sense in which " mathematics begins with Euclid ").

Belonging as it does to the series *II : Naturwissenschaften* of the History of Science edited by E. Rothacker, the present volume is concerned strictly with scientific mathematics as so distinguished, from the origins to early 20th Century. The two main aspects of history are kept in view, the one rooted in the relationship of individual man to the community and environment, the other in manifestations of complex entities (cultures, peoples, races etc.), which have interesting analogies in modern cosmological theories. A judicious combination of the two aspects is regarded by the authors as an essential part of their task in the present work.

There are two sections. The first, by O. Becker, of 80 pp. with a 16-page bibliography, is on Ancient Mathematics (one third pre-Greek and two-thirds Greek). The second, by J. E. Hofmann, deals first (25pp.) with Eastern Mathematics (Hindu, Moslem, Chinese and Japanese), then (112 pp.) with Western Mathematics from the break-down of Antiquity, ending up with a briefly outlined, essentially up-to-date history (8 pp.) of the History of Mathematics, an unusual and noteworthy feature, and a 116-page bibliographical index, of incalculable value.

With that, any adequate subject index, apart from the table of contents, was out of the question, owing to size and cost limitations. Although it may be hoped that such exigencies will disappear in future versions, the omission has in

so slim a volume two advantages. Greater prominence is thus given to the factual basis here insisted on, viz. the printed and written evidence ; and a reader interested in the evolution of a special field or in isolated facts is obliged to see the context of history of mathematics as a whole before sorting out the details he particularly wants ; and in this instance the trouble of making one's own index is not excessive and is well repaid. As an example, the much-discussed Newton-Leibniz controversy, a speciality of J. E. Hofmann, is set back among innumerable quarrels and arbitrary and secretive acts by these and other mathematicians, in that and other periods, and becomes subsidiary to the even greater effect of friendships, friendly interrelations, corporate solidarities, and the preservation and transmission of material by these means. Other subject matter often overlooked is, for more modern times, the teaching of mathematics, the introduction of the decimal system, and (unfortunately somewhat out of context on p. 230) mathematical dictionaries and Encyclopedia articles.

The reader is trusted to use his intelligence and to follow up clues, without being teased by enigmas, —a treat in this journalistic age. One must be rather sharp (unless forewarned) to see that the " Reziprokenpaare " on p. 28 are pairs of numbers whose product is 60, and that the semi-colon on p. 34 divides the integral part from the fractional in a sexagesimal notation for numbers. But in the main the effort needed to fill out the scaffolding is moderate.

This opinion must be qualified in one respect for English readers. The facility for complex phraseological structure in the German language makes it possible to close-pack information in one sentence in a way unparalleled in modern scientific English. With an average command of German, some points will be missed on first reading and misunderstandings, especially in J. E. Hofmann's style of writing, are likely. An English rendering, though no easy matter, would be a benefit to many.

R. C. H. Young.

Les Machines à Penser. By Louis Couffignal. Pp. 155. 1952. (Les Editions de Minuit, Paris)

The advent of the first large automatic calculating machines gave rise to a spate of colourful stories in the popular press seasoned indiscriminately with such terms as " electronic brains ", " thinking robots ", etc. Following protests from academic circles this fashion has now subsided, but there is still a singular lack of informed literature on the subject for the layman to read.

The author of this book is to be commended for attempting to produce an account of the principles of automatic computing without shrinking from the inevitable delicate questions raised by allusions to " mechanized thought ". He is careful to point out how and to what extent, however slight, each particular type of machine described might be said to think.

Unfortunately the actual descriptions of machines are altogether too sketchy to be properly understood by those without prior knowledge. Thus, although eight pages are spent in establishing the mathematical basis of a continuum, in the next nine pages the whole theory and practice of the differential analyser are dismissed, finishing with a faulty diagram which the reader is left to follow for himself.

It is a pity that the only contemporary digital machine mentioned is that being built by the author, for this machine is not quite typical of the many in existence. Indeed the factual descriptions seem to have been hastily compressed in order to devote the second half of the book to a flood of highly imaginative speculations on the nature of thought which, although perhaps of individual relevance in an ultimate discussion between experts in the field, might be taken collectively in their present context by some readers as more typical of informed opinion than is the case. Few of those who have worked on recent

machines would be as optimistic as the author regarding the possibility of constructing a machine to imitate human thought. S. GILL.

Basic Arithmetic. Bks. 1*a* and 1*b*. By T. G. DAFFERN. Pp. 88, 65. 3s. 6d. each. 1951. (Basil Blackwell)

Here are two books which will delight the heart of the teacher who is wondering how to interest a set of backward children at the bottom end of the Secondary Modern School. Mr. Daffern, who has had experience with this type of child, has set out to cater for his special needs in a practical and comprehensive manner. The mechanical processes taught are quickly followed up with easy applications so that the child does not lose sight of the connection between theory and practice.

These books, which are intended for 1st yr. pupils, are divided into sections dealing with number and money in Bk. A and Weight, Capacity and Length in Bk. B. The work is so arranged however that revision is incorporated into many of the exercises, or units, as they are called. Each unit is clearly labelled 1, 2, or 3, so that the quick, average or slow child may work at his appropriate rate.

The lay-out of the books is excellent—they are attractively illustrated throughout and are a refreshing change from the usual " books of sums " which, of necessity, many of the Modern schools are using. Where reading might present difficulty the sketch accompanying the unit will probably be an assistance, but the language throughout is simple and well within the compass of an eleven year old child. Unfortunately there is confusion between the use of the words " pence " and " pennies " and between " halfpence " and " halfpennies ", but no doubt this will be corrected in the next edition.

Naturally these books are in no way replacing the Mathematics teacher, but in helping to supplement his work they are invaluable. I am looking forward to the publication of the rest of this series. Y. B. G.

Made to Measure. Bks. 1 and 2. By J. H. EBBUTT. Pp. 47 each. 1s. 3d. each. 1951. (A. & C. Black Ltd.)

In view of the fact that in the Primary School reading often presents difficulty to the children, these books, which are intended for the 7–9 year old, are probably more useful to the teacher than the pupil.

The lay-out of the work is clear and easy to follow. The aim throughout is to provide, in a colourful and interesting way, exercises in the use of ruler and scissors. This is achieved in Bk. 1 by a series of carefully graded block patterns based on the square and rectangle and necessitating measuring in inches and half inches. The intention is for the child to read the printed instructions, cut out the shapes suggested in gummed paper, and then to copy the design with the aid of the diagram outlined on the opposite page. Bk. 2 progresses to the use of quarter inches and besides more elaborate patterns, includes the drawing and cutting out of objects such as a pillar-box. Unfortunately where for accuracy the use of a set-square is obviously involved, no instructions are given.

Informal work of this type is usually more effective as a class or group activity under the guidance of the class teacher. So often the value of this basic mathematics can be lost to the individual child because of the obstacles encountered in the text. For this reason, " Made to Measure " is recommended as a reference book for the Staff and not as a text-book for the class.

Y. B. G.

Graded Examples in Arithmetic and Trigonometry. By L. E. LEFÈVRE. Pp. 120. 5s. 1952. (A. & C. Black)

This book contains 120 papers of ten questions each. The author's intention

is that eight questions from each paper shall represent one hour's work. The papers are divided into three sections: questions 1–6 form a straightforward set on a single topic; questions 7 and 8 are revision questions; 9 and 10 are harder revision questions, which could be made either alternative to 7 and 8 or extra questions.

The subject matter covers all topics in the two subjects likely to appear in ordinary levels of the G. C. E., with the exception of "non-numerical" trigonometry. The questions appear to be well graded and are carefully chosen. Answer books are supplied by the publishers to *bona fide* teachers.

F. J. T.

Geometry and the Imagination. By D. Hilbert and S. Cohn-Vossen; translated by P. Nemenyi. Pp. ix, 357. $5. 1952. (Chelsea Publishing Company, New York)

A reprint by Dover Publications of the original work was reviewed recently (Vol. XXXVI, p. 231), and all that remains is to commend this translation. Production is excellent, diagrams and printing clear, and the text lucid.

A glance down the index (twenty-five columns of it) reveals the breadth of range:— Annulus; Atomic structure; Automorphic functions; Bubble, soap; Caustic curve; Color problem; Density of packing, of circles; Four-dimensional space; Gears, hyperboloidal; Graphite; Lattices; Mapping; "Monkey saddle"; Table salt; Zinc. These are but a few of the topics brought before the geometer's view. The title invokes the imagination, and the text must surely capture it.

E. A. Maxwell.

Optics. By Sir Isaac Newton. Reprint of the 4th (1730) edition. Pp. cxv, 406. $1.90; cloth $3.95. 1952.

Science and Hypothesis. By H. Poincaré. (Rep.) Pp. xxvii, 244. $1.25; cloth $2.50. 1952.

Science and method. By H. Poincaré. (Rep.) Pp. 288. $1.25; cloth $2.50. 1952. (Dover Publications, New York)

Newton's *Opticks* was reprinted in 1931 by Messrs. Bell & Sons, with a foreword by Einstein, and an illuminating introduction by Sir Edmund Whittaker, at a time when the 19th-century view that the wave theory was "right" and the corpuscular theory hopelessly "wrong" had ceased to have that hard certainty characteristic of the century. But the chief interest of the reprint lies perhaps in the opportunity it gives us to see Newton at work, to understand why Einstein says of him: "Nature to him was an open book, whose letters he could read without effort".

Poincaré's books are still a sheer joy to read. Moving with perfect ease and grace over the whole field of pure and applied mathematics, Poincaré was perhaps the last of the universalists; he was certainly not far from the first. The translator's note to *Science and Hypothesis* is signed W. J. G., which may serve to remind members of the Mathematical Association of one whose steadfast devotion to the advancement of mathematics never flagged.

The Dover reprints, making available so many out-of-print classics, can be had in two forms, one paper-bound, the other in cloth. This is a great convenience for those who wish to stretch modest purses to the fullest extent.

T. A. A. B.

A Concise History of Mathematics. By Dirk J. Struik. Second revised edition. Pp. xix, 299. Paper $1.60; cloth $3. (Dover Publications, New York)

Dover Publications are issuing some of their scientific books in cheap paper-bound editions at prices from 1.25 to 1.90 dollars. This concise History is well

printed on good paper and firmly stapled: the book will not stay open by itself, but shows no tendency to fall apart which seems more important. It contains a great number of illustrations, mostly good; portraits of Gauss and Poincaré are sad exceptions. There are useful bibliographies at the end of each chapter, referring the reader both to original sources and to second-hand books and articles. A particularly welcome feature is the reproduction of pages from original works—although this is led ad absurdum when 19th century editions of (e.g.) Leibnitz' or Euler's works are given in *facsimile*. Although the story is necessarily condensed, the author has rightly thought it essential to sketch the general historical background; this is, however, the least successful part of the book, both in selection and in outlook—in particular our author dislikes the Greek slave-owning, leisure-class aristocracy who earn disapproval for things which are said to do credit to, say, the Indians—a little social history is a great evil. Altogether this is a profitable book, well balanced, omitting the ever repeated anecdotes, but giving a number of often neglected facts (as for example Fibonacci's proof that a certain cubic cannot be solved by quadratic radicals); it will be welcomed by teachers who are encouraged by modern examination syllabusses to teach some history of their subject. A. P.

Many-valued Logics. By J. B. Rosser and A. R. Turquette. Pp. 124. Fl. 12. 1952. Studies in logic and the foundations of mathematics. (North-Holland Publishing Co., Amsterdam)

The systematic study of many-valued logic was initiated by Emil L. Post, in 1921, in his " Introduction to a general theory of elementary propositions ". In the following twenty years, although certain special extensions of classical logic were introduced, the general theory of many-valued logic attracted scant attention, but in the last decade there have been great technical developments, due in no small measure to the efforts of the authors of the present book. A satisfactory *interpretation* of many-valued logics has, however, not yet been found, nor, as yet, has any useful application for them been discovered. It is not even clear that there are any problems which many-valued logic can solve which cannot be solved by the ordinary two-valued logic. Rosser and Turquette stress the need for research into these questions and deplore as premature some recent attempts to resolve the conflict in modern theories of physics by means of a three-valued logic.

Following the pattern set by the current treatment of classical logic, many-valued logic is presented both by means of truth tables and in the form of an axiomatic formal system. In place of the familiar connectives, *and, or, not, implies*, and the truth values *true, false* of two-valued logic, Rosser and Turquette introduce a finite list of statement functions $F_r(P_1, ..., P_{a_r})$, $1 \leqslant r \leqslant b$, and M truth values denoted by the first M integers. The first S of these values are called designated and the remainder, undesignated. An asserted statement is characterised by having a designated truth value, and one that is denied, an undesignated value (so that in a rough and ready way increasing truth values may be associated with decreasing likelihood). With each statement function $F_i(P_1, ..., P_{a_i})$ is associated a truth value function $f_i(p_1, ..., p_{a_i})$ such that if each p_r is a positive integer between 1 and M then so is f_i. The role which negation plays in a two-valued system is taken in a many-valued logic by the $M - S$ statement functions $J_r(P)$, $S+1 \leqslant r \leqslant M$, which are such that $j_r(p)$, the value of the truth function corresponding to $J_r(P)$ is designated when, and only when, the value of p is r.

As an example of the general theory, the special case of a logic with two statement functions $F_1(P_1, P_2)$, $F_2(P_1)$, with associated truth functions $max(1, p_2 - p_1 + 1)$, $M - p_1 + 1$, is considered in detail. Many-valued disjunctions and conjunctions are introduced by defining PVQ as $F_1(F_1(P, Q), Q)$

and $P \& Q$ as $F_2(F_2(P) \vee F_2(Q))$. Statement functions $J_k(P)$ with associated truth functions $j_k(p)$, satisfying $j_k(k)=1$ and $j_k(p)=M$ for $p \neq k$, are constructed. Defining the operator $\bar{P}$ as

$$J_{s+1}(P) \vee J_{s+2}(P) \vee \ldots \vee J_M(P)$$

it is shown that $\bar{P}$ and $\bar{P} \vee Q$ have truth value properties which are generalisations of the familiar two-valued negation and implication.

Fundamental to the systems considered are the so-called standard conditions imposed on many-valued truth functions; the functions $\&(p, q)$, $\vee(p, q)$, $\supset(p, q)$, $\sim p$ and $j_k(p)$ are said to satisfy standard conditions if (by analogy with the classical connectives) the value of $\&(p, q)$ is designated if, and only if p, q are both designated ; $\vee(p, q)$ is undesignated if, and only if, p, q are both undesignated ; $\supset(p, q)$ is undesignated if, and only if, p is designated and q, is undesignated ; $\sim p$ is designated if, and only if, p is undesignated and, finally, each $j_k(p)$ is designated if, and only if, $p=k$.

The axiomatisation of many-valued logic is effected through the seven axioms :

1. $Q \supset (P \supset Q)$;
2. $(P \supset (Q \supset R)) \supset (Q \supset (P \supset R))$;
3. $(P \supset Q) \supset ((Q \supset R) \supset (P \supset R))$;
4. $(J_k(P) \supset (J_k(P) \supset Q)) \supset (J_k(P) \supset Q)$;
5. $\Gamma_1{}^m (J_1(P) \supset Q) Q$;
6. $J_1(P) \supset P, \quad 1 \leqslant i \leqslant S$;
7. $\Gamma_{k=1}^{\ \beta} J_{p_k}(P_k) J_f(F_i(P_1, \ldots, P_{\alpha_i})), \quad 1 \leqslant i \leqslant b, \quad f=f_i(p_1, \ldots, p_{\alpha_i})$;

(where $\Gamma_1{}^n A_i B$ denotes

$$A_n \supset (A_{n-1} \supset (\ldots A_2 \supset (A_1 \supset B)) \ldots)$$

and the rule of inference that Q is an acceptable statement (theorem) if P and $P \supset Q$ are acceptable.

It is shown that if the truth-value functions $\supset(p, q)$, $j_k(p)$ satisfy standard conditions the axiomatic determination and the truth-value determination are equivalent in the sense that the class of theorems determined by each is the same, and this equivalence characterises the systems studied in this book, by contrast with such systems as Lewis's *strict implication*, or *intuitionistic logic*, which are not deductively complete, that is to say, are not capable of deriving all the theorems which have designated truth-values. The metasystem in which the equivalence proof is carried out is of course the classical two-valued logic.

The development of many-valued logic is carried out to the level of the predicate calculus. In place of the familiar universal and existential quantifiers we meet a finite class of quantifiers

$$\Pi_i(X_1, \ldots, X_m, P_1, \ldots, P_n),$$

containing m individual variables and n statements (or predicates) so that each quantifier may bind several variables at once, and combine several statements in the process. Many-valued predicate calculi are formulated both on a truth-value basis and by axiomatisation, and the two formulations are proved equivalent, under certain standard condition assumptions. The

method by which the deductive completeness is proved is a generalisation of Henkin's elegant proof of completeness of the two-valued predicate calculus. Many-valued predicate calculi are shown also to satisfy the Löwenheim-Skolem theorem.

Although the book is clearly not intended for beginners, the authors have taken great pains to make it easy to read. There is a glossary of the main symbols introduced, and page references to each axiom, hypothesis, derivation rule, lemma and theorem, together with a very complete and well-fashioned index. The typography is of the fine quality we have learned to expect in books in this series. R. L. GOODSTEIN.

Enzyklopädie der Mathematischen Wissenschaften. Vol. II, section 1, part i. Mathematische Logik. By H. HERMES and H. SCHOLZ, MUNSTER. Pp. 82. DM 8.20. 1952. (Teubner, Leipzig)

This part of the encyclopaedia is devoted entirely to the familiar two-valued logic of classical mathematics. It opens with a list of periodicals specialising in foundation studies (the foremost of which is the *Journal of Symbolic Logic*) and a short book list which includes both nineteen century works as well as current texts on mathematical logic. As one expects from the Head of the leading European centre for research in mathematical logic, Professor Scholz has produced an account which, despite the limitation to 82 pages, is impressively comprehensive and complete to the end of 1950.

The plan of the article follows familiar lines. We meet first the propositional calculus, the calculus of *and, or, not, implies*, for which both the truth-table and the axiomatic formulations are described ; this is followed by the pure predicate calculus, which introduces " object " variables and the universal and existential quantifiers, leading to the predicate calculus with identity, which admits equations as concrete instances of predicates and by a suitable axiom make possible the substitution, one for another, of identicals.

The principal results described may be summarised as follows :

1. The propositional calculus may be formulated as a system of demonstrably independent axioms, which is demonstrably complete and free from contradiction.

2. The Löwenheim-Skolem theorem which says that a system of logic which admits a model (in the sense in which a set of numbers may be a model of an axiomatic geometry) necessarily admits a denumerable model (so that, for instance, if there is an interpretation of the system in the field of real numbers there is an interpretation in the field of rationals.)

3. Gödel's theorem which says that the pure predicate calculus is complete in the sense that every true formula is derivable.

4. Church's theorem which says that there is no mechanical decision procedure for determining all the true formulae of the pure predicate calculus (although there are such procedures for limited parts of the calculus).

5. Gödel's theorem that the predicate calculus with identity (fortified by suitable defining equations for addition and multiplication) admits true formulae which are not derivable.

In this connection we may mention a recent result of Tarski's that a suitably defined algebra of real numbers in which integral variables play no part is demonstrably complete (i.e. contains no insoluble problems).

The article concludes with an account of the logical paradoxes and their resolution by type theory or by means of Quine's method of stratification, and a section (contributed by G. Hasenjaeger) on Gentzen's ' inference logic " which is based on a relation $R(A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_m)$ between two

sequences of propositions $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_m$, which asserts that one of the B's follows from the totality of A's. Inference logic is set up as a formal system which is shown to be complete and syntactically free from contradiction.

R. L. GOODSTEIN.

Dictionary of Conformal Representations. By H. KOBER. Pp. xvi, 208. \$3.95. 1952. (Dover Co., New York)

This dictionary, by a distinguished analyst, was originally compiled during the war for use in war-time research. It consists of five parts whose headings are, Linear and Bilinear transformations; Algebraic functions and z^α for real α; $w=e^z$, $w=\log z$ and related functions; Schwarz-Christoffel transformations representable in the form of elementary transformations; Higher transcendental functions. There is also a short but useful bibliography.

The subject of conformal representations has been developed, and is being developed, by mathematicians working in different fields often with the aim of solving some special problem in applied mathematics. For this reason it is both peculiarly difficult to present the subject in a coherent and comprehensive manner and at the same time particularly desirable that some such presentation should be available. Although the author specifically disclaims any pretensions to completeness, he has in fact succeeded in so arranging his material that all the conformal transformations that are of a general importance or significance, apart from some of those which depend upon the higher transcendental functions, appear in this book, and yet it is not unduly bulky.

This achievement has been made possible by the use of a complex variable notation for the various curves that are of interest in the representation and by the use of a large number of excellent figures. The equations of curves in terms of complex numbers have mostly been known for a considerable time, but their use here has enabled the author to develop the subject thoroughly and has led him to fill in some of the more obvious gaps in our knowledge. He has obtained general results that are new, even in the case of the well known and elementary bilinear transformation. By the use of figures in the two complex planes, the author has been able to express the fundamental properties of the representation in a succinct, vivid and easily assimilable form.

The two possible methods that could have been adopted as a basis for the classification of the mappings are either to classify them in terms of the nature of the mapping function or in terms of the nature of the curves that map into some given curve. Of these the author has chosen the first as being on the whole more systematic than the second. To complement this arrangement a topological subject index is included which classifies the transformations in the second manner.

This book as its title indicates, is a reference book and not in any sense a text-book, so that no proofs are given and the information contained consists entirely of statements of results. It appears to be the first book of its kind on this subject and may well remain the standard work for many years. For teachers of complex variable theory this book is useful as it can be used as a source book for examples. For research workers in complex variable analysis and in those many branches of applied mathematics that use conformal representations, this book should be invaluable since it collects so much information that was formerly not readily available. But the greatest use of it will perhaps be made by physicists and engineers who will find in it the solutions of many of their problems involving the two-dimensional Laplace equation.

H. G. EGGLESTON.

Advanced Calculus. By W. Kaplan. Pp. xiii, 679. $8.50. 1952. (Addison-Wesley Press, Cambridge, Mass.)

The subject matter of this book includes most topics which usually fall under the heading of advanced calculus. However, there is more than the usual emphasis on applications and on physical motivation.

A basic review of what is to be assumed in the remainder of the book is given in an introductory chapter. Vectors are introduced at the outset and used wherever possible throughout the book. In the portion dealing with integration numerical methods receive special attention; improper integrals and infinite series are treated similarly and are co-ordinated at the end of chapter 6. There is a chapter on Fourier series and orthogonal functions and a concise treatment of differential equations including the use of numerical methods. The last two chapters deal respectively with functions of a complex variable and partial differential equations. The former extends to some 128 pages and develops the subject from the beginning to the stage where it may be used in Hydrodynamics; some of these applications are in fact considered.

A high level of rigour has been maintained throughout. Certain deeper theorems required in order to deal with the finer points are not however proved in the text but reference is given to the more advanced treatises.

Each chapter concludes with a large number of examples and a list of books for supplementary reading. P. M. H.

Elementary Coordinate Geometry. By E. A. Maxwell. Pp. 288. 18s. 6d. 1952. (Oxford University Press)

This book, at sixth form level, is intended for those who hope to become mathematical specialists.

The subject is developed from the beginning. Co-ordinates and straight lines are discussed in the first two chapters but in order to be able to deal more conveniently and compactly with certain aspects of the subject to follow later, chapter III is devoted to a discussion of the solution of linear equations and determinants. The remainder of the book is written, however, so that although determinants are actually used in the text they are not essential to the arguments involved.

Chapter IV has as its title " Introduction to Analytical Methods ". The emphasis here is on the methods available and these are given with reference to a particular typical curve so that a thorough survey of the way in which the subject is to develop is obtained.

The portion of the book dealing with conics can be divided into two parts in the first of which the methods of analytical geometry are used. The curves, apart from the circle, are defined parametrically and appear in the following order; parabola, rectangular hyperbola, circle, ellipse, hyperbola, which is chosen since the circle although most familiar geometrically is less attractive analytically. In the second part, where the focus-directrix definitions are given, the methods of pure and analytic geometry are used and are closely interwoven, the emphasis being on the geometry of the figures. Between these two parts three typical curves possessing a point of inflexion, a cusp, and a double point are discussed parametrically and a chapter is devoted to envelopes in which the locus of the line $lx+my+n=0$ where l, m, n, are functions of t is considered. The book concludes with a chapter on polar co-ordinates in which the polar equations of the conics are established.

There are plenty of examples mostly taken from papers set by the Oxford and Cambridge Schools Examination Board; sections which can be omitted on first reading are clearly indicated and on several occasions valuable hints have been given for the teacher himself.

Throughout the book the author has borne in mind the requirements of the teacher and has produced a work which will obviously be used effectively in the classroom. P. M. H.

The Methods of Statistics. By L. H. C. TIPPETT. Fourth Revised Edition. Pp 395. 38s. 1952. (London: Williams and Norgate Ltd.; New York: John Wiley and Sons Inc.)

The first edition of this book appeared at a time (1931) when Fisher's book, "Statistical Methods for Research Workers", first published in 1925, was becoming known as an indispensable text for experimental workers, mainly at that time in the biological sciences, because of the new methods advocated to secure maximum information from relatively small samples of observational data. Tippett's book covered very much the same ground, and was successful because of a clarity and simplicity of style which made an immediate appeal. Revised editions appeared in 1937 and 1941, but the present revision is of a more fundamental character, and merits re-examination of the book as a whole. It is much increased in size, and, unfortunately, in price, but it should be stated at once that it is a good book, and will find a definite place as an introductory text among the large number of treatises now appearing on the subject of statistics. The author's clarity of style is still there, and despite his modesty in the preface concerning the ramifications of the subject on which he is not an authority, we have here a well-selected and authoritatively stated selection of reading matter which may be said to be essential knowledge to the statistician.

A straightforward listing of contents is hardly necessary, since there is much that must nowadays appear in such a text. But it will be of assistance to potential readers to indicate how far the author goes, and what particular topics of a more advanced or specialized character he deals with. The usual sampling distributions and tests are discussed: on the question of inference the methods of Fisher and of Neyman and Pearson are given, with a discussion of maximum likelihood, and of confidence limits and fiducial probability. The ordinary analysis of variance is expanded to deal with multi-factor cases and the question of orthogonality; correlation analysis contains sections on biological assay and ranked data; the multiple regression chapter deals with discriminant functions, and the penultimate chapter, on non-linear regression, deals with orthogonal polynomials and the analysis of time series.

The last chapter consists of a collection of problems of practical application, although it should be said here that such a description could be given to much of the contents of the earlier chapters. The author's own work lies in the industrial research field, and he is fully aware of the fact that a method, or a test, can only justify its inclusion in a working manual if it provides a wanted tool. The chief topics dealt with in the final chapter are sampling and the principles of experimental arrangement. In all cases enough is said to convey useful knowledge to the reader; if more is wanted there are nowadays specialized texts on experimental designs, sampling, probit analysis, rank correlations and so on, and references for further reading are supplied, together with the necessary tables and charts for the computations described in the book.

J. WISHART.

The Science of Chance. By HORACE C. LEVINSON. Pp. 303. 30s. 1952. (Faber, London)

This non-mathematical description of probability theory and of its applications to statistics is written for the general reader. Its author is an American consultant on business statistics.

The book consists of two parts. In the first, entitled Chance, the basic ideas

of probability theory are developed and are applied in detail to various games of chance, including poker, roulette, craps, and bridge. The author evidently has much first-hand knowledge of all these games, and brings both experience and reason to his aid in persuading his readers that, in the long run, gambling pays only the organisers of casinos and other gambling institutions. The mathematics of this part of the book is simple, but its detailed application to the computing of probabilities of runs in roulette, and of different kinds of hands in the card-games, would be tedious to any reader who is not a keen exponent of the games analysed.

In the second part of the book, entitled Statistics, the author deals descriptively with the mean and the standard deviation of a frequency distribution, and with the Normal distribution. He shows how these concepts can be applied to various practical problems drawn mainly from the fields of advertising and business accounting; he also illustrates some of the common fallacies that arise from the misapplication of averages.

For the general reader the book is a well written, sound, and readable introduction to probability theory and elementary statistical methods, but the mathematical reader will find its exposition laborious. The publishers claim too much when they say that it " discusses what has to date been learned about the science of probability ". B. C. Brookes.

Mathematics. A Second Course. By M. F. Rosskopf, H. D. Aten and W. D. Reeve. Pp. 365. 24s. 1952. (McGraw-Hill)

In the review of the First Course I mentioned that the amount of Geometry in the Course was negligible. The comments then made must be modified in the light of the now available Second Course. If the surprise of the First Course was that it contained practically no Geometry, that of the Second Course is that it contains little else. The only other matter consists of some revision examples in Algebra at the end of each of the 12 chapters, some mensuration arising out of the Geometry and a few pages of Trigonometry. The treatment of the subject is very different from that in England. It differs on so many points that it is not easy to deal with and comment adequately on all the issues which are raised.

While allowing for differences between the educational systems in America and here, a number of criticisms are unavoidable. Nevertheless, it would be very surprising if there were nothing to learn from the methods of other people. I would draw attention to the symbols for similar triangles ($\sim$) and congruent triangles ($\simeq$; this means " is similar and equal to "). I have used the former at my school for a dozen years; in the *Gazette* for May 1939 E. G. Phillips mentioned that he always used this symbol for similar triangles. The American symbols for congruence and similarity are also in general use in, at least, Germany. In view of the fact that the symbol for similar triangles has very wide currency and has stood the test of time, we could adopt it here with advantage seeing that we have no generally accepted symbol. Congruence of triangles is confined to the cases *SAS*, *SSS* and *ASA*. The *AAS* and *RHS* cases are not mentioned. Throughout the book, the reader is expected to take the words " Circle O " to mean " Circle of centre O ". If this is meant to be a convention, it has never been defined. It seems to conform to the general looseness of wording to which the book is prone. For instance, we have: " Summarize the four minimum conditions for determining a plane " (p. 115); " What is the locus of points equidistant from a point? " (p. 230); "What is the locus of points equidistant from the faces of a pyramid? " (p. 230); " The area of a polygon is 256 " (p. 277). There is a similar " looseness " in the development of the text. For example, in the theorem on contact of circles, the construction given is: " Draw the common tangent MN " (at the point

of contact). Yet contact of circles and the existence of a common tangent at the point of contact is nowhere discussed.

There are certain essentials one expects of a text-book. It should be possible for a pupil to revise any matter and he should be able to use the book as a ready source of reference. He can do neither with this book since the enunciation of a theorem is usually not given in either the general or the particular form. Nor is there a proof which the pupil can consult. Here is a typical example of how the book deals with a theorem. The heading is " Study Assignment 74 ".

" *Given data :* Circle O and O' intersect ; line OO' is the line of centres. AB is the common chord. [figure supplied.]

" *Problem :* To discover a relationship which is always true.

" *Suggestions for the proof :* Use locus to prove the theorem. Where is O' with respect to A and B? Where is O with respect to A and B? What is OO' with respect to A and B? What is the locus of points equidistant from two points? Place the statement of the theorem in your notebook under ' Theorems we have proved '."

The extent to which pupils are able to make correct use of the hints will vary considerably. This will also vary with the clarity of the " hints ". Consider, for instance, the theorem of " Study Assignment 60 ". This deals with the angle in the alternate segment. The text goes :

" *Given data :* Circle O with chord AB and tangent BC.

" *Problem :* To find the measure of $\angle ABC$.

" *Suggestions for the proof :* Draw AX parallel to BC [chord AX is implied] An inscribed angle is formed. [In previous work ' inscribed angle ' is used for angle subtended by an arc at the circumference.] Compare the angle and its arc with the angle ABC and its arc. The statement of the theorem can be summarized as follows : An angle formed by a tangent and a chord drawn to the point of contact is measured by ? its arc. Place the statement in your notebook under the heading ' Theorems we have proved '."

Will this be clear enough to enable pupils to provide their own proofs? I doubt it. Of the many points arising out of these illustrations there is not space to deal with more than two. One is the question of the extent to which notes should be used in school mathematics. In my view these should be very few and confined to such things as essential formulae. The more this is exceeded the greater is the waste of time. The pupil using the book under review is expected to write anything up to 150 enunciations requiring some 600 to 700 lines. Remembering that these are not ordinary easy composition, they will take up to about 25 lesson periods. On the basis of two periods of Geometry per week, this corresponds to a term of wasted effort. The second point is that the teaching—in Geometry of all subjects—should, in the main, be the business of the teacher, not of the text-book. The text-book should provide the right material in carefully graded and well-arranged sequence.

An important feature of the book is the applications of reasoning processes to events in everyday life. Sets of exercises on non-geometrical applications are given in every chapter and these are almost as long as those on geometrical applications. Pupils are made to feel that Geometry and the methods of Geometry have an essential place in a general education. The geometrical riders are usually very simple and cannot compare either numerically or in difficulty with those in an English text-book. There are several interesting suggestions for constructing simple surveying instruments which are well worth the consideration of teachers.

S. I.

Dimensional Analysis. By H. E. Huntley. Pp. viii, 158. 20s. 1952. (Macdonald)

This book, which is suitable for students in the last year at school or in the first two years at the University, is a welcome addition to the literature on Dimensional Analysis. The early chapters cover the normal ground with a wealth of examples from such widely spread topics as " surface tension ", " the blueness of the sky " and " the period of a tuning fork ". One omission, however, is that there is no reference to the impossibility of the variable x in any function $f(x)$ having dimensions, i.e., $f(x)/f(0)$ has no dimensions. The student only too often obtains an expression such as sin t, where t is time, without realizing that t must be multiplied by an angular velocity.

These are followed by chapters in which distinctions are made between unit lengths in three directions and this addition to the fundamental quantities is shown to have many useful applications to problems in which velocities and accelerations occur. The applications to finding the period of torsional oscillations and to finding the velocity of a sphere falling under gravity in a viscous fluid are very ingenious.

The remaining chapters are devoted to discussing thermal and electrical quantities. In the earlier chapter emphasis is laid on the fact that each physical quantity can be allocated a definite dimensional formula, but in the chapter on electricity the only reference to the possibility of having a unique dimensional formula for an electrical quantity is to Professor Cramp's suggestion of adding Q, the quantity of electricity, to the fundamental quantities mass, length and time. As a matter of fact, if a velocity is always introduced in the formulae for curl $\mathbf{E}$ and curl $\mathbf{H}$ (either the velocity of light or unit velocity), there is no need to give dimensions to K or μ or even to introduce any fundamental quantity in addition to M, L and T. An objection to this procedure is that $\mathbf{E}$ and $\mathbf{H}$ then have the same dimensions but possibly this difficulty could be removed by an application of vector unit lengths as in the earlier chapters.

Electrical units and dimensions are however dealt with quite as satisfactorily as in any other text-book and the examples on this topic as well as the other examples throughout the book are admirably chosen. For this reason especially the book can be thoroughly recommended. It is attractively printed, but, as in many other modern text-books, a magnifying glass is sometimes needed to read the fractional indices. H. V. L.

Mathematics for Students of Technology : Junior Course. By L. B. Benny. 2nd edition. First year : pp. vii, 120, viii. 6s. 6d. Second year : pp. 121–282, xiii, 7s. 6d. 1952. (Oxford University Press)

First published in 1929, the first edition has had nine impressions. The author states that he has made very few changes, except to correct errors in the answers, to enlarge the chapter on percentages and to add a set of miscellaneous examples. He is very wise, because these well known books are firm favourites with teachers in Secondary Technical Schools and are an admirable introduction to the Mathematics of the National Certificate Courses.

The books are clearly printed and pleasant to handle, while the examples are carefully graduated.

Twenty three years after the first publication, not all teachers will agree with the author's views expressed in 1929. For example, he is bitterly opposed to using logarithms (and the slide rule apparently) until the theory of indices is grasped, the negative number is delayed until long after the introduction of Algebra, the straight line graph is too divorced from the idea of proportion and seems to be in a " pigeon-hole " of its own, while science masters will

desire an earlier introduction to trigonometry. This does not matter provided that the order of treatment is in the hands of an experienced teacher.

Having reached a heavy chapter on factorising trinomials, is it not a pity that the reader is not to have some view of the promised land of the Quadratic equation?

With regard to geometry, twenty three years ago Mr. Benny had the role of a prophet, but until recently his advice fell on the deaf ears of the disciples of "Practical Mathematics". Present day thoughts on the reintroduction of Geometry to our Technical Students place Mr. Benny's books at the head of of Teachers' booklists.

Although I feel that the Trigonometry section is dated and too heavy, I never pick up these books without admiring the author's cunning introduction to Algebra.

He suggests that these books might be used in Modern Schools and in the general education courses of Technical Colleges for those of 15–16 years of age. It is feared that such pupils might find the books too technical and the examples too difficult. A. J. L. AVERY.

Mathematics for Students of Technology. Senior Course. By L. B. BENNY. 3rd edition. First year: pp. vii, 1–268, xv, iii; 7s. 6d. Second year: pp. 269–469, xxx, vi. 8s. 6d. 1952. (Oxford University Press)

For twenty-five years Mr. Benny's book has held a place of honour among the treatises on "practical mathematics". In this new edition the text is substantially unaltered, and it is a measure of the value of the book to say that we do not feel that alterations were called for, though the reviewer is still not convinced that the engineer should be asked to differentiate a^x as an introduction to the exponential.

A new set of miscellaneous exercises and the addition of three recent National Certificate examination papers make the book even more useful than it was in its earlier editions. T. A. A. B.

Mathematics for Telecommunications. Vol. I. By D. F. SPOONER and W. H. GRINSTED. Pp. xiv, 341. 10s. 6d. 1952. (English Universities Press)

This volume covers the field of the First Year City and Guilds syllabus in Mathematics for Telecommunications. The nine chapters deal with elementary algebra; graphs; logarithms and slide rule; formulae; equations; further graphs; geometry; trigonometry and mechanics.

The authors have set out to show the student the application of mathematics to telecommunication engineering. The early reference to degrees of accuracy is very useful and the chapters on graphical representation are in line with the engineer's conception of mathematics.

The arrangement of the book is unusual in that the authors have dealt with formulae manipulation, evaluation of formulae and simple equations in that order. This does not seem to be desirable from a student's viewpoint, particularly as the book is intended for home study as well as a class text-book.

In chapter 3, logarithmic calculations are performed with "powers of 10" (i.e. retaining the base) but in the next chapter the more usual tabular form is adopted. Apart from introductory purposes, the latter form would have been adequate since the aim is to use mathematics as a tool. "Standard form" is used to obtain the characteristic.

Some of the explanation is rather tedious, partly due to the order of the chapters. As a result an important later chapter on trigonometry is too cramped. This subject, including angles of any magnitude, is disposed of in 25 pages.

The book assumes some previous knowledge of symbolical expression and

the use of directed numbers. Drill exercises have been cut to a minimum, which detracts from the value of the book for class work.

The authors have attempted the difficult task of correlating the requisite mathematics with the technical side of telecommunications and have achieved a considerable amount of success.

Although the book is not recommended as a textbook for First Year Electrical National Certificate students it would be a useful supplement to their lecture notes and should receive the attention of their lecturers. F. JEEPS.

Workshop Practice. By A. E. PEATFIELD. Pp. 207. 6s. 1952. (English Universities Press)

In attempting to cover a wide range of topics, the author has found it necessary to treat some important items very briefly. Grinding and Milling are, however, worthy of more attention than knurling and drill nomenclature.

The student will find much descriptive matter on equipment in the latter part of the Machine Shop Work section with but little instruction in the art of using such equipment.

The book otherwise covers the field more fully than one would expect at such a modest price and the author is to be congratulated on his use of line diagrams throughout. D. I. ROBSON.

Differential Calculus 5th edition. Pp. xii, 460. 6 Rs. 1951.
Analytical Solid Geometry 7th edition. Pp. viii, 272. 5Rs. 1952.
For B.A. and B.Sc. students of Indian Universities. By SHANTI NARAYAN. (Doaba House, Delhi)

The first of these books has reached its 5th edition in 9 years, and it can be assumed that it meets all demands. Is it the reviewer's fancy to discern the influence of G. H. Hardy in the opening chapter on real numbers, which are well and clearly dealt with? Or is this only to be expected from an author of the race which taught the rest of the world how to count?

The rest of the development of topics follows the usual lines, unfortunately so in the case of the logarithmic and exponential functions where the author might well have followed Klein and Hardy, with a note of explanation to his students after the manner of Nunn and Fletcher. Mr. Narayan has in this more excuse than English writers since 1930, for they still fail to show the beginner a clearer approach to the theory of logarithms.

The course followed is comprehensive and thorough, and there is a good chapter on curve tracing. The author has a talent for clear exposition, and is sympathetic to the difficulties of the beginner. The same remarks can be applied to the second volume : the subject here is much less exciting, but the book is competently and clearly written and arranged.

In both books the standard of printing and the quality of the paper are not good even by the lowest of English standards. Perhaps the American comment about most current English book production might be the same. Answers to examples, of which there are good and ample selections, are given, but why no index?

A further point, quite unconnected with these particular books, and yet prompted by their perusal : would we, to whom a foreigner's English is so often a matter for amusement, be quite so rapid in our studies if our texts were printed in Tamil or Urdu? Certainly Mr. Narayan's command of the English is excellent. It seems that one of the most valuable gifts of the British Raj to India is a lingua franca. Europe on both sides of the curtain is in need of such. Our own young scientific or mathematical specialists, grumbling over French or German or Latin as additions to their studies, would do well to consider their Indian confreres, with English to master before their technical education can begin. J. E. B.

National Society for the Study of Education: 50th Yearbook, Part II: The Teaching of Arithmetic. Edited by NELSON B. HENRY. Pp. xii, 302. Cloth 26s.; paper 21s. 1951. (University of Chicago Press; Cambridge University Press)

Here's richness: 300 pages about teaching arithmetic by an impressive array of authorities. Great argument there is, about it and about, and inevitably a good deal of repetition. No one could envy Mr. Henry his task, which is well done, and both printers and publishers have backed him up with an equal display of efficiency.

What is abundantly clear throughout is that the teaching profession in the U.S.A. is greatly concerned with the presentation of arithmetic in a "meaningful" way to all kinds of pupils, gifted and otherwise. There is a praiseworthy undercurrent of feeling in all the articles that children would be less bored or hopeless with numbers if they were better taught on the fundamentals. It is odd, though, to find no mention of Maria Montessori, although the influence of her ideas is apparent. But would she, I wonder, have approved of the passive regarding of films as an aid to teaching? It is even more strange to find in none of the analyses of the principles of calculation any clear distinction between partition and quotition. An excerpt, quoted below, from suggestions for research underlines this curious omission.

Mr. Thiele of Detroit, in a lucid article on work in the middle grades, hits the nail squarely on the head:

"One of the problems of the teacher, then, is to guide, direct and stimulate children to make number generalisations."

Yet even this for the middle grades is understatement: it is the teacher's main concern.

The book is worth careful study by all teachers of elementary mathematics, wherever they have their being, but I cannot help feeling that the Society has rather overdone things. The conscientious young teacher, after ploughing through it, might feel slightly dazed by the volume of advice and precept. Some of the suggestions for research suffer from an excess of zeal, and provoke the wish that Mr. Nelson might have had greater powers in wielding the blue pencil. This one, for instance, never would be missed:

"The extent to which learning is accelerated or impeded by the language of the teacher or the instructional materials. For example, is the pupil who understands what is meant by "Divide the candy bar into thirds" temporarily blocked in his learning by "Divide 3 into 15" when confronted by the situation $3\overline{)15}$? If he were as intelligent as his teacher should be, might he not reply "I can divide 15 into 3s, but it is not possible to divide 3 into 15."

Zeal, all zeal, Mr. Easy!

On finishing the book, Hilaire Belloc's lines seem particularly apt:

"These facts should all be noted down
And ruminated on
By every boy in Oxford Town
Who wants to be a Don."

J. E. B.

British Association Mathematical Tables, Vol. X. Bessel Functions, Part II: Functions of Positive Integer Order. Prepared by W. G. BICKLEY, L. J. COMRIE, J. C. P. MILLER, D. H. SADLER and A. J. THOMPSON. Pp. xl, 255. 60s. 1952. (Published for the Royal Society at the University Press, Cambridge)

This handsome volume is the last of the series of British Association *Mathematical Tables*, as responsibility for the work on mathematical tabulation formerly undertaken by the British Association for the Advancement of Science was transferred to the Royal Society in 1948. It continues the tables

published in 1937 in the sixth volume of the series (*Bessel Functions, Part I : Functions of Orders Zero and Unity*). It is noted in the Preface that the volume " has been in the minds of the British Association Mathematical Tables Committee for very many years, and its preparation has used a large portion of the energy and resources of that Committee since its reorganisation in 1929 ". It forms a brilliant termination of a distinguished series, and one entirely worthy of the beautiful printing of the Cambridge University Press.

The tables relate to the Bessel functions $J_n(x)$, $Y_n(x)$, $I_n(x)$ and $K_n(x)$. The greatest value of the order n is 20. The values of the argument x run up to 25 for J and Y, and up to 20 for I and K ; the interval in x is always either 0·01 (but never for $x > 10$) or 0·1. For further description, the tables fall into two groups.

Tables I–IV (pages 2–179) give values to about 8 figures (exactly 8 decimals in the case of J) for $n = 2(1)20$, values for orders zero and unity having already been provided in Part I. While the functions $J_n(x)$ are tabulated directly, as also are the $Y_n(x)$ for the larger values of x, auxiliary functions are given in the other cases ; thus for the smaller values of x the tabulated functions are $x^n Y_n(x)$, $x^{-n} I_n(x)$ and $x^n K_n(x)$, while for the larger values of x they are $e^{-x} I_n(x)$ and $e^x K_n(x)$. This mode of tabulation enables ordinary methods of interpolation to be used throughout. Second differences, modified when necessary, are given wherever they are appreciable. In many regions these suffice ; where a residual fourth-difference correction is needed for full accuracy, the modified second differences are italicized.

Tables V–VIII (pages 180–255) are basic tables to about 10 figures (exactly 10 decimals in the case of J) for $n = 0(1)20$, with interval 0·1 in x throughout, and without differences. The functions $J_n(x)$, $Y_n(x)$, $I_n(x)$ and $K_n(x)$ are themselves tabulated, rather than auxiliary functions. Users are warned in the Introduction that interpolation in these basic tables is not easy. Interpolation by differences being hardly feasible, it is desirable to make use of special properties of the functions in question. Three ways of performing the feat are described and numerically illustrated.

The values in the basic tables should be correct to within half a unit in the last place, whereas errors up to 0·52 final units are possible in the interpolated values in the first four tables.

An interesting experiment in presentation is made in the basic tables of $Y_n(x)$, $I_n(x)$ and $K_n(x)$, which vary enormously in order of magnitude. In such cases the commonest arrangement is to print the successive groups of (say) ten significant figures vertically under one another, giving when necessary an indication of the number of zeros separating them from the decimal point. In the present instance, however, it was decided to keep digits in the same decimal place vertically under one another. As all ten significant figures are sometimes more than thirty places in front of the decimal point or more than forty places behind it, the columns in the more extreme cases lean over like trees in a gale. As is remarked in the Introduction, the " curve " formed by the leading digits is itself a rough graph of the function on a logarithmic scale.

In addition to information about the arrangement and construction of the tables and about processes of interpolation, the Introduction contains a bibliography, a summary of notations, and a highly detailed list of formulae which will be invaluable. The acknowledgements show that the volume is the work of many hands ; all are to be congratulated on the result of their efforts. The importance of Bessel functions in many fields of application requires no emphasis. The reviewer has only one lament, that " integer " is used as an adjective instead of " integral ", both in the title and in the text.

A. Fletcher.

A Short Table for the Bessel Functions $I_{n+\frac{1}{2}}(x)$, $(2/\pi)K_{n+\frac{1}{2}}(x)$. Prepared on behalf of the Mathematical Tables Committee of the Royal Society by C. W. Jones. 20 pp. 6s. 6d. 1952. (Published for the Royal Society at the University Press, Cambridge)

This pamphlet gives tables for Bessel functions of imaginary argument and half-odd-integral order. The chief quantities tabulated are :

$$x^{-\nu}I_\nu(x), \quad (2/\pi)x^\nu K_\nu(x), \quad \text{for } x=0(\cdot1)5,$$
$$e^{-x}I_\nu(x), \quad (2/\pi)e^x K_\nu(x), \quad \text{for } x=5(\cdot1)10,$$

where $\nu=n+\frac{1}{2}$ and $n=0(1)10$. Almost everywhere, seven or eight significant figures are given. Modified second differences are provided. There are also a few auxiliary and supplementary tables.

There is no doubt that the functions involved have been inadequately tabulated in the past. The tables under review have been constructed and published in response to requests from workers in theoretical physics and quantum chemistry. They will also be useful in other fields, and their publication is very welcome.

It is interesting to note that the tables were set on a Vari-typer in the office of the Royal Society, and printed by photolithography. The result, though inevitably inferior to printing from ordinary type, is uncommonly good of its kind, and quite adequate for practical purposes.

A. Fletcher.

Numerical Analysis. By D. R. Hartree. Pp. xiv, 287. 30s. 1952. (Geoffrey Cumberlege, Oxford University Press)

This book is based on a course of lectures given by the author for several years in the Mathematical Laboratory of the University of Cambridge. It assumes no previous knowledge of the theory and practice of systematic numerical work, but is not on that account an elementary text-book. Some idea of its scope may be obtained from the titles of its twelve chapters : Introduction ; the tools of numerical work and how to use them ; evaluation of formulae ; finite differences ; interpolation ; integration (quadrature) and differentiation ; integration of ordinary differential equations ; simultaneous linear algebraic equations and matrices ; non-linear algebraic equations ; functions of two or more variables ; miscellaneous processes ; organization of calculations for an automatic machine. For several of these topics, the book does not by any means cover the existing knowledge and, as the author points out, some of the later chapters could each be expanded into a volume. Within its declared limitations, this is an excellent book, and the status and reputation of its author are such as to guarantee that it is completely reliable and authoritative.

There may be a little disappointment amongst those who knew this work to be in preparation that a volume bearing the comprehensive title *Numerical Analysis* should turn out not to be the definitive treatise that might have been hoped for. The subject has been taking shape over the past twenty years or so, partly as a development of the Calculus of Observations of former days, partly from the analysis of physical problems giving rise to differential equations ; drawing also upon the techniques of mathematical table making, upon the use of commercial calculating machines, and the intricate devices of the computer's art. At present, much of the knowledge is scattered in miscellaneous papers, in academic text-books on finite differences, in volumes of tables, or not written down at all. There is a clear need for this to be drawn together into a series of books.

The treatment adopted in the present volume is avowedly a practical one. " The subject of numerical analysis is concerned with the science and art of

numerical calculation, and particularly with *processes* for getting certain kinds of numerical results from certain kinds of data. . . . Although the end is a numerical result, algebra and analysis are involved in the development of these processes. . . . But the algebra and analysis must be aimed at providing or establishing *practical methods of obtaining numerical results*; otherwise it may be elegant mathematics, but is not a contribution to numerical analysis." The inevitability of mistakes is pointed out, and in many places it is shown how important it is to incorporate adequate checks into the calculations. All this is a welcome antidote to the view that " putting numbers into formulae " is just a technician's job—instead of being, if it is done properly, a skilled process which may dictate the choice of the formulae themselves, or even modify the underlying mathematical calculation.

Amongst the good points of Professor Hartree's book are: (i) numerous worked examples, showing desirable layout and typical mistakes, sometimes with a diagrammatic description of the order in which steps are carried out in such processes as the numerical integration of differential equations; (ii) suitable prominence given to the central difference operator δ, with the logical use of the symbol δx for the quantity which has been variously called h, w, etc. Aspects with which the reviewer is less satisfied are: (i) a rather " thin " account of mathematical tables, which are so often the means of transmitting the results of numerical analysis to others; (ii) a set of exercises which is not particularly extensive.

C. W. Jones.

Über die Klassenzahl Abelscher Zahlkörper. By H. Hasse. Pp. ix, 190. \$6.48. 1952. (Akademie Verlag, Berlin)

This book, as is suggested by its title, is a very technical one, and assumes a considerable knowledge of algebraic number theory. It is concerned with the discussion of the well-known formula for the number of classes of ideals in an Abelian algebraic number field. The formula has usually been considered as the final object in its derivation. It is by no means obvious from the formula that the class number is either positive or an integer, or has any arithmetical significance; and so much remains to be done even when the formula is given. Further, it is not very suitable for calculation and so there remains the problem of reducing it to a more convenient form.

Hasse gives an interesting and comprehensive account of the subject and of the relevant known results and their relation to each other. He concludes with some extensive tables that are important and have authentic significance.

This book is obviously a most valuable one for all who wish to study the subject and its development.

L. J. Mordell.

Theory of Matrices. By S. Perlis. Pp. xiv, 237. \$5.50. 1952. (Addison-Wesley, Cambridge, Mass.)

This textbook deals with the parts of matrix theory which centre round the various reductions to canonical form. It should be useful both to the students specializing in pure mathematics and (in providing the mathematical background) to those concerned mainly with the applications. With the non-specialists in mind, the author has taken pains to make his account readable and interesting, to give his proofs in careful detail and point out pitfalls, and (in most places) to make clear his motives. Several applications to linear differential equations are given as short illustrations of the algebraic theory. The material is economically and purposefully arranged, and the parallels between different theories duly emphasized. The methods tend to be concrete and practical.

An outline of the contents will indicate the scope of the book. Chapters 1–3 centre round equivalence ($A \to PAQ$), which is treated by means of elementary

row and column transformations. This leads to the usual theorems about linear equations, rank, inverses, etc. Vector spaces come in Chapter 2, determinants (for the first time) in Chapter 4. Chapter 5 deals with congruence and conjunctivity. Chapters 6–8 develop the elementary divisor theory for matrices with polynomial elements and then pass, via a division algorithm for these matrices, to similarity ($A \to P^{-1}AP$). Chapter 9 deals with similarity reduction to diagonal form, in particular by a transforming unitary matrix. Chapter 10 deals with linear transformations and so gives a motive to Chapters 8 and 9. The first five chapters are perhaps the best written, and Chapter 9 the worst, though it contains a lot of material. In the latter it would perhaps have been easier and more illuminating to prove that an arbitrary matrix is unitary similar to a triangular matrix and thence to deduce the theory of normal matrices. Chapter 10 is rather an anti-climax and might have been absorbed into the earlier chapters.

My general opinion is that this is a good and useful book, without being an outstanding one. It is well printed and bound, though expensive for British students.

Correction : Theorem 5–25 is wrongly stated, as the function

$$(x_1 + x_2)^2 + (x_1 - x_2)^4$$

shows. G. E. Wall.

The Theory of Lattices. By B. C. Rennie. Pp. 51. 6s. 6d. ; 10s. 1951. (Forster and Jagg, Cambridge)

This little book is essentially the author's Ph.D. thesis and consequently is rather technical in character. Its purpose is to develop an approach to lattice theory differing from the usual one by its emphasis on the topological properties of a lattice. Using a modification of MacNeille's " completion of a lattice by cuts " and several definitions of convergence, eight intrinsic topologies are defined in terms of lattice structure only and all the relations between these topologies are established. The remaining chapters discuss " cup- " and " cap- " continuity, metric lattices, Boolean algebras, simply ordered sets, the lattice of closed sets, lattice groups, Banach lattices and almost periodic sets. Some unsolved problems are also mentioned.

The book is well printed and carefully written but although it is self-contained, the reader will find many passages difficult to follow unless they are read in conjunction with Birkhoff's book, to which numerous references are made. Had the author allowed himself twice the number of pages, the book would have been much more readable. D. E. R.

Foundations of Combinatorial Topology. By L. S. Pontryagin. Pp. xi, 99. N.p. 1952. (Graylock Press, Rochester, N.Y.)

This little book is an extremely valuable addition to the literature of algebraic topology. It can, in fact, be recommended very strongly to the new graduate who wishes to prepare himself for research in topology and to the third-year man who offers combinatorial topology as a special subject in his Finals. The book's suitability may be judged from the fact that it is essentially an English translation of a course of lectures given by the distinguished author at Moscow University.

The book covers the homology theory of polyhedra, the proof of the topological invariance of the homology groups (here called Betti groups) of a complex, mappings of polyhedra into polyhedra and the induced homomorphisms of the homology groups, and the Lefschetz fixpoint theorem for mappings of polyhedra. The development proceeds smoothly (and rigorously) from initial definitions to the main results, the author allowing himself two digressions,

one to the proof of the embeddability of a compact metric space of n dimensions in euclidean space of $(2n+1)$ dimensions, and the other to a proof of Sperner's lemma, the topological invariance of the dimensions of a complex, and Brouwer's fixpoint theorem. The latter digression is rather quaintly inserted in the middle of the proof of the invariance of the homology groups, as a breather, presumably. The reviewer recommends the reading of § 10 after § 12.

The book contains no treatment of the fundamental group and covering complexes, of the homology theory of surfaces, or of Poincaré duality and intersection theory in a manifold. Even more serious—and more avoidable—is the omission of any examples, for which the author apologises. Nevertheless, the reviewer has no hesitation in describing this book as filling a need long felt by the supervisors of young research students.

On p. 22, the sets F_i may have diameter $\eta/6$, and the proof of Theorem 5 needs a slight adaptation to this possibility. On p. 31, line 9, it should be asserted of the group A that all its *non-zero* elements are of order m. On p. 55, line 6 from bottom, "ϵ-mapping" should be "ϵ-covering". Finally, the reviewer warns students against assuming that the presence of *Foundations of algebraic topology*, by S. Eilenberg and N. Steenrod, in the list of references indicates that it is a suitable text for one who has just completed his study of the present work. P. J. HILTON.

Elementary Analysis. By K. O. MAY. Pp. xviii, 635. 40s. 1952. (John Wiley, New York ; Chapman & Hall)

This volume, which was first published in 1950 under the title *Analysis, a Freshman Course*, is designed to provide "a unified treatment of material usually labelled algebra, analytic geometry, trigonometry, and introductory calculus. The book is directed to those who are going to use mathematics in advanced courses or in science, engineering, or business. Although it assumes as a minimum only one year of high school algebra and one of plane geometry, there is plenty of material for a year's work by students with more extensive preparation". There are fourteen chapters in the book, headed: Introduction, Logic, Number and Elementary Operations, Linear Functions, The Quadratic Functions, The Power Functions, The Exponential and Logarithmic Functions, Circular Functions, Analytic Geometry, Complex Numbers, Conic Sections, Polynomials, Algebraic Functions, Functions of Two Variables; of these only the third, the last (which deals with three-dimensional geometry) and the chapter on complex numbers contain anything which a mathematics specialist in one of our grammar schools would not have learnt within six months of completing a course in Additional Mathematics.

The treatment throughout is very detailed and elementary, and particularly so in the first half of the book. Thus, even on p. 92, we find such examples as "Find the decimal equivalent of 3/8" and "Express 0·03 as a percentage". But though elementary and discursive, the book is thoroughly sound and could profitably be read by anyone intending to continue the study of mathematics at a more advanced level. Unfortunately, for so elementary a book, its price in this country is high.

There are over six thousand exercises, worked and unworked, many of them of practical interest. Answers and tables are provided. R. W.

A School Arithmetic. II. By L. E. LEFÈVRE. Pp. xv, 123. Without answers, 4s.; with answers, 4s. 6d. 1952. (A. & C. Black)

This volume completes an O-level arithmetic course. The chapter titles are: Interest, Profit and Loss, Circles, Mensuration, Problems, Proportion, Stocks and Shares, The Use of Tables.

The book is written in the same racy style as Volume I, but the author combines it with a rare thoroughness of treatment. He does not shirk difficult or awkward questions, and explanations are generally quite adequate. It may be felt that the informal style is occasionally overdone, as when the Stephen Leacock story of A, B and C is quoted at length, or when ? is used as a symbol for the unknown in an equation. The choice of examples is excellent ; there are not too many and they are well graded. The author abhors learning by rote, and in fact gives no formula for simple interest. The variety of the examples used in the chapter on Proportion is astonishing, and the instructions and worked examples in the chapter on the Use of Tables are admirable.

The author disapproves of antilogarithm tables and square root tables, and has excluded them from the collection of printed tables in the book. This reviewer strongly regrets such a course of action and believes that many teachers, like himself, find such tables more convenient, quicker to use, and conducive to more accurate working than the use of another table " in reverse ". It seems a pity that the chapter on Tables has not been put earlier in the book, so that some of the examples, for instance in mensuration, could have been set with the use of logarithms in view.

The book closes with 30 pages of miscellaneous examples. Apart from a serious misprint on p. 272 (where the surface area of a cone is given as rl) and the use of over-small figures for powers of 10, the printing and layout are excellent. F. J. T.

Übungen zur Projektiven Geometrie. By Dr. Horst Herrmann. Pp. 169 mit 90 Bildern im Text und 4 Raumbildern. 17 Swiss francs bound ; 14 unbound. 1952. Lehrbucher und Monographien aus dem Gebiete der Exakten Wissenschaften, Mathematische Reihe, Band 18. (Birkhäuser, Basel)

This book is a collection of examples, and is not a text-book, but intended as a companion to one. The arguments given are rather closely knit and the development of the subject matter is carried through in a logical order, so that the book deserves to be read on its merits, and indeed a certain amount of continuous reading is necessary before the reader can obtain full advantage from it. The purpose of the book is to demonstrate the use and effectiveness of matrix methods in projective geometry and the result is a technical tour de force. The general use of matrix methods in this subject has been gaining wide acceptance over a number of years, but the author's applications, especially to configurations, go far beyond what is to be found in standard English text-books. The reviewer is unable to dispel a sneaking feeling that this limited approach does not quite come off, and that the elegance and power of the methods used is no adequate compensation for a certain lack of variety and the absence of synthetic methods.

However, no personal views should be allowed to stand in the way of this book being given the widest possible circulation. It will not interest all students of the subject, but it is one of the books that will greatly appeal to a few gifted students and provide a powerful stimulus. It should certainly therefore be in all libraries where it can fall into the proper hands. As a source of examination questions the book looks to be a gold mine, and professionals will look forward to tracing its influence in final degree examinations and the competition for the Administrative Civil Service over the next dozen years.

A striking feature of the book, in addition to the large number of diagrams in the text, is a set of four delightful stereographic diagrams of space configurations provided at the end of the book. These are viewed through the usual red and green spectacles, a pair of which is provided to fit neatly into a

pocket with the plates. Perhaps readers may be warned, from the reviewer's experience, that small boys will treat this part of the text with more enthusiasm than restraint, and that the spectacles are rightly designed only for fair and proper use. The lack of an index is a serious fault in a book which is intended as much for dipping in as for reading: it is hoped that the publishers will remedy this at the earliest opportunity. D. B. S.

College Geometry. By N. ALTSHILLER-COURT. 2nd edition. Pp. xix, 309. $4. 1952. (Barnes & Noble, New York)

This book provides a comprehensive discussion of the familiar geometry of the circle and the triangle, and ends with an account of the "recent" geometry of the triangle as investigated by Lemoine, Brocard, Tucker, etc. It covers the metrical geometry required to our Open Scholarship standard. Much stress—perhaps too much—is laid on construction problems and the reader is invited (or shown how) to construct triangles and circles from every conceivable triad of data. No mention is made of the harmonic property of the quadrangle—indeed a quadrangle is referred to *passim* as a quadrilateral—nor of harmonic ranges on a circle. Perhaps these are regarded as belonging to the field of Descriptive Geometry. There is, as the author admits, far more material than can be assimilated by the ordinary student, but a selection of topics can be made to provide an instructive course either for him or for the enthusiast.

The proofs and constructions are, in general, notable for elegance and economy, but it is a pity that no indication is given in the text of the relative importance of the theorems proved. Ceva's Theorem, for example, and "The reciprocal of the inradius of a triangle is equal to the sum of the reciprocals of the altitudes of the triangle" are given equal prominence, and the book would be more valuable if some distinction were drawn between those theorems essential to the development of the subject and those which—however interesting—are in themselves dead ends. One wonders, too, why the author was content with so pedestrian a solution of the problem of Pappus—through a given point on the bisector of a given angle to draw a secant on which the given angle shall intercept a segment of given length.

The book is excellently printed, with no observable misprints, and the diagrams are clear and easy to comprehend. The purist may find pleasure in the meticulous way in which, almost without exception, each theorem is given a general enunciation in which no element is referred to by name. The enthusiast will welcome it as a compendium of metrical geometry, and its utilitarian value is enhanced by an index which enables any topic to be quickly located, or the definition to be found of some term more familiar, perhaps, to the American reader for whom the book is intended.

W. J. H.

Introduction to the Theory of Games. By J. C. C. MCKINSEY. Pp. x, 371. $6.50. 1952. (McGraw-Hill)

This is one title in a series of books published in U.S.A. by the RAND ("*R*esearch *and* *D*evelopment") Corporation, a non-profit organisation "to further and promote scientific, educational and charitable purposes". Other books in this series bear such titles as "Soviet attitudes toward authority" and "Air War and emotional stress". It seems therefore best to state at the very beginning of the review of that book with which we are here solely concerned, that it is a genuinely mathematical work of very high calibre. It was written as a text-book for college courses and is very suitable to introduce this new and astonishingly lively branch of mathematics to readers who wish to

judge it on its scientific merits, apart from its applicability to warlike or economic competition.

The history of this field is rather remarkable. John v. Neumann started it in 1928 in a talk to the Goettingen Mathematical Society. He and O. Morgenstern published during the war their monumental *Theory of Games and Economic Behavior* and then things started moving, mainly in two American centres: at the Rand Corporation, and at Princeton University. The literature of the subject is rather scattered, and those interested were gratified to find in 1950 that *Annals of Mathematics Study*, No. 24, contained a great number of papers collected under the title "Contributions to the theory of games". By now this has also become old history. "Contributions to the theory of games, II," *Annals of Mathematics Study*, No. 28, is to be published and the book under review contains many references to proofs and arguments from it.

The theory of games does not teach us, of course, how to play any particular game well. It is rather a theory of those features of activities which arise when not all relevant factors are under one's control—hence the applications to economic behaviour. It receives its peculiar attraction from a curious philosophy which defines the notion of playing safe in a new way and builds on it a theoretical structure mainly in order to prove that such playing safe is possible and makes sense. The mathematician who has followed so far is then rewarded by most unexpected connections with other branches.

Mr. McKinsey's book is the first textbook of the subject. It starts off with such concepts as strategies, pay-off, saddle points and the Fundamental Theorem, viz.

$$\operatorname*{Max}_{x}\operatorname*{Min}_{y}\sum_{i}\sum_{j}a_{ij}x_iy_j=\operatorname*{Min}_{y}\operatorname*{Max}_{x}\sum_{i}\sum_{j}a_{ij}x_iy_j\quad(=v,\text{ say})$$

whatever a_{ij}, where x_i and y_j are restricted by $\sum_i x_i=\sum_j y_j=1$ and must be non-negative. The a_{ij} may be interpreted as the final payment to be made from player B to player A, if the former plays the game according to strategy "j" and the latter according to strategy "i". The x_i and y_j are relative frequencies with which the strategies are used in a long series of games. The change of strategies is made for safety. Methods are given to "solve the game", i.e. to find v (the "value of the game") and the x_i and y_j producing it. These relative frequencies are optimal, because if you use them, then your opponent cannot make your situation worse, even if he finds out about them.

With Chapter VII we reach games with infinitely many strategies, which are not yet mentioned in v. Neumann and Morgenstern's book. These games have been found of interest in statistical applications (Wald's theory of decision functions), but the subject is not simple and this book deals only with those games where the pay-off function has specific, more easily manageable, forms.

One chapter is devoted to the proof that the problem of solving a game can be transformed into that of finding non-negative values of n variables which satisfy m linear equations ($m<n$) and maximise a given linear form (so-called Linear Programming). Finally, games between more than two players and the v. Neumann-Morgenstern theory of non-zero sum games (where the total of all payments received by all players does not add up to zero) are dealt with. The last chapter is called "Some open problems".

Each chapter is followed by historical and bibliographical references and by a collection of exercises, without solutions. The book does not require more than a slight knowledge of analysis, though it does demand some maturity of mathematical thinking, as every such book should.

We do not expect (as yet) to introduce the theory of games into the syllabus of our secondary schools, but boys and girls might do worse than spend a few enjoyable hours with this book, if they can find it on the shelves of their school library. S. VAJDA.

Practical Descriptive Geometry. By HIRAM E. GRANT. Pp. vii, 253. 34s. 1952. (McGraw-Hill)

Third angle projection is used throughout this book, a feature which will appeal to many students in view of the gradual change to this projection which is taking place in this country.

This subject is always difficult to present and is usually of some trouble to the student. In this book exceptional skill and ingenuity has been shown and due to this, and the profuse illustrations, the book is above average standard. The practical illustrations are a good feature and help to dispel the rather theoretical air which many text-books on this subject possess.

A comprehensive index is provided, but no problems are set, since a special set of problems entitled "Practical Descriptive Geometry Problems" has been designed for use with this book.

The price is rather high for the average student. P. SHAW.

Intermediate College Mechanics. By D. E. CHRISTIE. Pp. xvi, 454. 59s. 6d. 1952. (McGraw-Hill)

This book written by an American professor provides an introduction to mechanics and uses the vector notation and treatment wherever possible. The treatment is essentially elementary and the subject matter ranges as far afield as the elements of elasticity, fluid mechanics, and the kinetic theory of gases.

The book is quite good on the whole with such entertaining examples as the "mechanics of the Bohr Hydrogen Atom". In my opinion, however, the theorems of Gauss and Stokes are out of place in a book of this standard, especially when their proofs involve throwing all mathematical rigour to the wind. No serious effect on the book would have resulted from their omission.

Numerous examples of a practical nature are solved in the text and many exercises are set for the reader. A partial list of answers to the latter appears at the end of the book followed by a very adequate index to the subject-matter. J. WILLIAMS.

An Introduction to the Theory of Differential Equations. By W. LEIGHTON. Pp. vii, 174. 30s. 1952. (McGraw-Hill)

Differential Equations. By A. L. NELSON, K. W. FOLLEY and M. CORAL. Pp. x, 299. 17s. 6d. 1952. (Heath, Boston; Harrap, London)

The title of Professor Leighton's book is an indication that although this is a text-book, providing a first course in differential equations, rather more attention than usual is paid to certain fundamental aspects. Theorems on the existence and uniqueness of a solution are stated as required and enable the author to make full use of particular solutions in obtaining general solutions; the importance of verifying the linear independence of particular solutions so used is also stressed. Four of the fundamental theorems are proved in an appendix. The method of variation of parameters is introduced early, with the linear equation of the first order, being preferred on account of its generality to the various other techniques for obtaining particular integrals. For the rest, the usual types of equation and methods of solution, up to solution in series, are considered, except that there is no mention of Clairaut's form and other equations of the first order but not of the first degree. The last two chapters deal with oscillation theory at a more advanced level, with a short introduction to characteristic functions (eigenfunctions) and expansion in a

series of orthogonal functions. The book is well-produced but, at the English price of 30s., somewhat lacking in substance. There are few physical applications and the short chapter on applications to mechanics extends to fifteen pages, of which seven are devoted to constant acceleration. The only misprint noticed is an obvious *dx* for *dy* in an equation used on p. 12, Exercise 3, and again on p. 15, Exercise 4.

The second book reviewed here is rather more in the style of English textbooks, introducing the usual methods of attack with plenty of exercises for the student. The range is somewhat wider, including partial differential equations of the first order, of which complete integrals are found by the methods of Lagrange and Charpit. A chapter on numerical methods includes, besides Picard's method of successive approximations, formulae for approximate numerical evaluation based on Taylor's series and on finite differences. At the end of the book there are four-figure tables of natural logarithms, trigonometrical and hyperbolic functions and a table of integrals. In the section on simultaneous linear equations there is no mention of the fact that when the equations are all of the first order, after solving for one dependent variable, one may eliminate the derivatives of the others and so obtain no superfluous constants. Another possible criticism is that after the solution of Legendre's and Bessel's equations, the exercises set for the student consist of verifications of orthogonal properties and other identities ; some indication might be given of their value, or familiarity with the new functions might be better acquired by solving specific examples of the differential equations with simple boundary conditions.

C. G. P.

An Introduction to Statistical Calculations. By J. Mounsey. Pp. xi, 351. 15s. 1953. (English Universities Press)

This book has been written by an experienced teacher of statistics primarily for students in commercial and technical colleges who are preparing for professional examinations. As the various syllabuses of these examinations include the subject matter of the G.C.E. syllabuses of statistics, this book will interest all who teach elementary statistics.

The title is exact. In his preface the author confirms that statistical *calculations* only are to be dealt with ; theory and explanatory comment find no place in the text. Though some understanding of statistical methods can be imparted and some examination successes can be achieved by drilling students in statistical computations, those who rely on this book alone might be led to apply uncritically the techniques it describes and remain unaware of their limitations. Nevertheless, if the teacher can provide the theory and point the warning finger where necessary, the wealth of its illustrative examples (over 200) and numerical exercises (nearly 600) makes the book an invaluable aid to class teaching.

The contents range from the computing of averages to the testing of the significance of the difference between two means (the t-test), of the correlation coefficient (the z-test), and of the ratio of two variances (the F-test). Short accounts of index numbers, the calculation of higher moments, analysis of variance, and quality control are also included. Answers to the exercises are given, and those tested by the reviewer were found to be accurate. All the statistical tables needed are included. Misprints are remarkably few, though in places (e.g. p. 230) the printer has been embarrassed by the small page size.

At its price, and within the author's self-imposed limits, the book is the most detailed and comprehensive that has yet appeared. It is a welcome addition to the few books suitable for the teaching of elementary statistics.

B. C. Brookes.

Notes on the Teaching of Statistics in Schools. By B. C. BROOKES. Pp. viii, 80. 5s. 1952. (Heinemann)

When the Royal Statistical Society prepared its *Report on the Teaching of Statistics* (see *Math. Gazette*, XXXVI, No. 317, pp. 227–8, September 1952) the committee concerned had before it, according to the Foreword to the present booklet by Prof. E. S. Pearson, some notes prepared by Mr. Brookes on the questions under discussion. These make Section I of the book, which is thus on the general curriculum. It is intended for those teachers with no academic training in, or practical experience of, the subject. Many, in fact, of Mr. Brookes' examples are from the science laboratory, and he says " to make the discussion realistic it must be conducted by the teacher who was in charge of the laboratory experiments. . . . The results of some class experiments should be analysed as a whole rather than be scattered . . . in individual notebooks. . . . By this means, the teacher of chemistry avoids some otherwise inevitable arithmetic and yet gains more than he loses in trying to explain the quantitative facts. . . ." I think this implies an improbable attitude on the part of the science colleagues of most teachers of mathematics. Of course, as Brookes says, " in schools where the teacher of arithmetic is also responsible for the teaching of some science or geography, it will be easy for him ". But in view of the type of example chosen by the writer—determination of the internal resistance of a Daniell cell, the individual lung capacity, 2N solution of HCl mixed with 0·1N solution, or even burette readings—it seems a pity that there are not more examples from the mathematics lessons themselves. After all, most of us get our class to obtain values of π (as the ratio of a circumference to the diameter, or as the ratio of the area, by counting squares, to the square of the radius), or of a sine or tangent (as the ratio of sides of various right-angled triangles), or even of a conversion factor (as, for example, from lengths measured in inches and in centimetres) so that we have plenty of examples, right up through the school, of differing estimates of the same value from our various pupils. It seems a pity also that more use is not made of the median and of the cumulative frequency curve at these earlier stages. On the other hand, there are some good suggestions, such as the idea of using the average of ranges of four as a measure of the scatter, the chess-board scheme for displaying the results of throwing two dice, the plea for the wider use of such terms as ton-mile and man-hour, and the experiment about the increasing size of sample and its effect on the estimate of a proportion.

Some details may prove to be rather awkward for the non-mathematical colleagues. Such are, I think, the subtlety about *through* and " through " points (why not say " among "?), the " plotting " of p. 25 and the " recording " on graph paper of p. 39, the two regression lines of a scattergram, and the catches in connection with moving averages. What are the units of the deaths on p. 27?' Is a " rogues' gallery " of bad and misleading charts good pedagogy?

On the whole, however, I think it will prove that the course suggested will be, as I indicated in my review of the *Report* itself, far beyond the reach of the child of 13 or 14 in the non-grammar-school secondary school.

Section II is entitled " Teaching notes on the G.C.E. syllabuses in statistics ". It supplements usefully the two books referred to—Yule and Kendall, and Weatherburn (the latter reviewed in the *Gazette*, XXXI, No. 294, p. 125, May 1947). The suggestion (p. 59) about a practical confirmation of the rule about the variance of the sum of two independent quantities is interesting. I am not clear about the practical example on weighted sums of traffic on two roads. It is perhaps a pity that though reference is made to Arithmetical Probability paper, no use is made of Poisson Probability paper (the expense of repeatedly supplying such sheets can be reduced if the plotting is done by

small coloured buttons instead of pencil), whilst both in this section and in Section I use is not made of the source of many Poisson distributions (as for example, from the time intervals between successive passers-by as observed from a park seat, or of arrivals in a bus queue or at a quick-service counter) as an alternative to getting information about telephone calls from the office boy. This section can be cordially recommended to those who are working with Sixth Forms for G.C.E. statistics syllabuses.

FRANK SANDON.

Associated Measurements. By M. H. QUENOUILLE. Pp. x, 242. 35s. 1952. (Academic Press, New York; Butterworth, London)

In the *Mathematical Gazette*, XXXV (No. 314), pp. 287–288 (December 1951) a review was published of *Introductory Statistics* by this author. In some ways the present book is a sequel to that work, though it is not so referred to in the author's introduction. It is an interesting volume, and includes much that is not to be found in the majority of text-books. It is not stated how the book came to be written, but we note that Mr. Quenouille writes it from Yale University; the earlier book was written when he was still at Marischal College, Aberdeen.

The book is in four sections. Section A—Graphical Analysis—consists of three chapters, graphical investigation, estimation, and testing (pp. 1–48). There are a number of attractive short cuts included: six tables at the end are useful in applying the various procedures dealt with, though the source of them is not usually stated. Section B—Numerical Analysis (pp. 49–100)—has another three chapters, on linear, multiple, and curvilinear association respectively. This deals with correlation and regression, and involves reference to orthogonal polynomials. Tables vii–xii of the appendix are relevant to this section, the first three of these being reproduced from *Introductory Statistics*. The third section—Rapid Estimation and Analysis (pp. 101–150), of three chapters—deals, *inter alia*, with grouping, quantiles, punched card procedures, and, as elsewhere in the book, the question of the rejection of extreme observations, which is considered from various points of view. The last section—Analytical Complications (another three chapters, pp. 151–213)—considers transformations, including the probit transformation, time series analysis, with serial correlation and Markoff processes of various orders and effects of trends, and multivariate analysis, including the method of principal components (with factor analysis and canonical correlations). At the end of the book is a useful bibliography of eight pages, practically all of British or American authors. There are also 13 tables, those already referred to with one of the χ^2 distribution. It may be pointed out that Tables 7.2*a* and 11.5*a* (pp. 105 and 174) in the text might more appropriately have been included in the Appendix of Tables. The whole is rounded off by a satisfactory four-page index.

The book will probably prove to be a useful handbook for those who have to turn to various specialist procedures for particular purposes or who need to make rough estimates by short-cut methods. Many of these are not, as far as I am aware, to be found in any other of the text-books commonly available in this country. It is well written and produced. Misprints are few, and the mathematics kept as simple as possible, matrix notation being introduced only where considerable advantages are to be obtained. It is a pity that such an attractive volume can not be put on sale at a lower price.

FRANK SANDON.

Wind-Tunnel Technique. By R. C. PANKHURST and D. W. HOLDER. Pp. xviii, 702. 57s. 6d. 1952. (Pitman)

Applied Aerodynamics has always been be-devilled by more factors than can be taken account of in a single comprehensive theory, but never so much

as at present. Man now faces a challenge from Compressibility (abetted by Viscosity—of course) that can be met only by the firmest of alliances between the theorist and the experimenter. The day when the former could be of service, whilst pretending that Compressibility and Viscosity did not exist, is rapidly passing, and we are now facing a most unholy combination of the two. The only answer to such an alliance is to consolidate our own Entente Cordiale, for it is through experiment that new assumptions are checked, new theories compared and, ultimately, new concepts formulated.

For these reasons alone, *Wind-Tunnel Technique* by two members of the Aerodynamics Division of The National Physical Laboratory deserves the attention of all who are associated with any aspect of Aerodynamics. It is astonishing when one considers the number of books on this subject, that this book has no rival in this country. It has closed a deplorable gap most admirably and consists of an expert, yet balanced, resumé of the diverse aspects of Wind-Tunnel Design, Research and Experiment. Sufficient detail is given for a proper appreciation of each technique to be obtained, but not so much as to be wearisome to the non-specialist. A selection of leading references concludes each chapter, these bibliographies being invaluable, as information on this subject is widely scattered through numerous reports, primarily the R. & M.s of the Aernonautical Research Council, in this country. When will something be done about these latter, with their ten year time lag before publication in indexed volumes? The authors have attempted to chart a way through them with a numerical and date index, and have conveniently tabulated the lists, monographs, and indexes. This is a typical sample of the useful and unexpected information to be found in the appendices, which include numerous tables of commonly used functions of M and other quantities used in Aerodynamics, British and American aerofoil series notations, together with a complete list of French and German Aerodynamic symbols with their British equivalents.

The clear type, layout, and diagrams make the book a pleasure to the eye, and it must surely remain the standard work on its subject for many years to come. D. A. J.

Men of Mathematics. By E. T. Bell. Reprint in two volumes. Pp. 646. 2s. 6d. each part. 1953. Pelican Books, A 276, 277 (Penguin Books, Harmondsworth).

Bell's lively and stimulating collection of mathematical biographies should be in every school library. Penguin Books must be warmly thanked, not only for giving us this cheap edition at the price of an ounce of tobacco, but also for recognising that mathematics is neither the esoteric playground of an eccentric minority nor a collection of amusing puzzles.

The schoolboy, the ordinary citizen, even the mathematician, who would learn something about the personalities of the subject, can not do better than begin with these volumes. Bell's facts are sound, his style is racy, his enthusiasm infectious, and his opinions, whether we accept them or not, are at any rate not insinuated in any half-hearted deprecating fashion but set down clearly and forcefully, often with a humour gently malicious. Informative, exuberant, provocative, and all for five shillings! T.A.A.B

Elementary Integration By C. E. Kemp. Pp. 7. 1s. (The Bursar, Reading School, Berks.)

This little pamphlet is not a table of integrals, but a set of brief notes on methods of dealing with integrals likely to be encountered in the school. If the pupil has had a thorough grounding in standard forms and basic methods, these notes would form a valuable aid to memory. T.A.A.B.

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